UNIVERSITÀ DELLA
VALLE D'AOSTA
UNIVERSITÉ DE LA
VALLÉE D'AOSTE

Dipartimento di
Scienze Economiche e Politiche
Département des
Sciences Économiques et Politiques

Convex along lines functions and abstract convexity. Part II

Giovanni P. Crespi
Ivan Ginchev
Matteo Rocca
Alex Rubinov

Working Paper n. 2

Ottobre 2012

I Working Paper di Scienze Economiche e Politiche hanno lo scopo di favorire la tempestiva divulgazione, in forma provvisoria o definitiva, di ricerche scientifiche originali. La pubblicazione di lavori nella collana è soggetta a referaggio e all'accoglimento del Comitato editoriale. I Working Paper sono disponibili all'indirizzo: www.univda.it.

Convex along lines functions and abstract convexity. Part II.

Giovanni P. Crespi[*] Ivan Ginchev[†] Matteo Rocca[‡] Alex Rubinov

The paper deals with the problem to characterize the abstract subdifferentiability and the abstract convexity of a function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ with respect to the min-type elementary functions $\mathcal{L}_k$, concentrating to the case $k = n$. The cases when $k \geq n + 1$ have found a satisfactory solution in [5], so the case $k = n$ is in some sense crucial. Following a constructive approach to the posed problems, there are distinguished abstract subdifferentiable functions which admit abstract subgradients of special form, and similarly abstract convex functions being the upper envelope of abstract affine functions of special form. This leads to the definitions of $\mathcal{L}_n^0$-subdifferentiability and $\mathcal{H}_n^0$-convexity. The $\mathcal{H}_n^0$-convex functions are convex-along-lines (CAL), so the CAL functions attract our attention. The paper is a continuation of [1], where the case of positively homogeneous (PH) functions is studied. Here the investigation is extended to non PH functions.

1 Introduction

Abstract convex analysis, grown after the monographs of Pallaschke, Rolewicz [2], Singer [9], and Rubinov [5] to a mathematical discipline with its own problems, aims to generalize the results of convex analysis to abstract convex functions. Its importance is due mainly to its close relations to global optimization. One of the problems in abstract convex analysis is the characterization of abstract subdifferentiability and abstract convexity of a function f. In this paper we deal with the problem to characterize the abstract subdifferentiability and the abstract convexity of a function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ based on the classes of min-type abstract linear functions $\mathcal{L}_k$ and min-type abstract convex functions $\mathcal{H}_k$ ($\mathcal{L}_k$-subdifferentiability and $\mathcal{H}_k$-convexity), concentrating to the case $k = n$ (recall that $\mathcal{L}_k$ is defined as the set of the functions $\ell : \mathbb{R}^n \to \mathbb{R}$ being minima of k linear functions and $\mathcal{H}_k$ is obtained by adding constants to the functions of $\mathcal{L}_k$). The cases when $k \geq n + 1$ have found a satisfactory solution in [5], so the case $k = n$ becomes crucial. The paper is a continuation of [1]. While in [1] we confine the study to positively homogeneous (PH) functions, here we extend the investigation to non PH functions.

Actually, like in [1], we introduce the class of the $\mathcal{L}_n^0$-subdifferentiable functions as a subclass of the $\mathcal{L}_n$-subdifferentiable functions, and the class of the $\mathcal{H}_n^0$-convex functions

[*]Université de la Vallé d'Aoste, Facoltà di Scienze Economiche, Strada Cappuccini 2A, 11100 Aosta, Italy, e-mail: g.crespi@univda.it.
[†]University of Insubria, Department of Economics, Via Monte Generoso 71, 21100 Varese, Italy, e-mail: vechnig@yahoo.com.
[‡]University of Insubria, Department of Economics, Via Monte Generoso 71, 21100 Varese, Italy, e-mail: mrocca@eco.uninsubria.it.

as a subclass of the $\mathcal{H}_n$-convex functions, and characterize the functions $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ being $\mathcal{L}_n^0$-subdifferentiable (Theorem 8) and the functions $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ being $\mathcal{H}^0$-convex (Theorems 12 and 13). In particular these results show that the property of f to be convex-along-lines (CAL) is necessary for f to be either $\mathcal{L}_n^0$-subdifferentiable or $\mathcal{H}_n^0$-convex. Hence, as the title shows, we concentrate on CAL functions. The obtained characterizations are obviously related to the general problem, which still remains open, to characterize the $\mathcal{L}_n$-subdifferentiable functions and $\mathcal{H}_n$-convex functions ($\mathcal{L}_n^0$-subdifferentiability implies $\mathcal{L}_n$-subdifferentiability, and $\mathcal{H}_n^0$-convexity implies $\mathcal{H}_n$-convexity). For this (and only this) reason we consider the classes of $\mathcal{L}_n^0$-subdifferentiable functions and $\mathcal{H}_n^0$-convex functions of some importance.

This paper was planned earlier as a continuation of [1]. The sudden passing away of Alex Rubinov delayed the plans. Still, the remaining authors are glad to finish it, taking on them the responsibility for possible weaknesses.

2 Preliminaries

Let $\mathbb{R}_{+\infty} := \mathbb{R} \cup \{+\infty\}$, where $\mathbb{R}$ is the set of the reals. For each $x \in \mathbb{R}^n \setminus \{0\}$ consider the ray $R_x = \{\alpha x : \alpha \geq 0\}$ and the line $R^x = \{\alpha x : \alpha \in \mathbb{R}\}$. Given a function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ and a point $x \in \mathbb{R}^n \setminus \{0\}$, we denote by $f_x : R_x \to \mathbb{R}_{+\infty}$ and $f^x : R^x \to \mathbb{R}_{+\infty}$ the restriction of f respectively to R_x and R^x. The function f is called convex-along-rays (CAR) or convex-along-lines (CAL) if for any $x \in \mathbb{R}^n \setminus \{0\}$ the function f_x or f^x respectively is convex. The function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is called positively homogeneous (PH) if $f(\alpha x) = \alpha f(x)$ for all $x \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}_+ := [0, +\infty)$. Here we follow the convention that $0 \cdot (+\infty) = 0$. So, if f is PH, then $f(0) = 0$, and consequently $0 \in \operatorname{dom} f := \{x \in \mathbb{R}^n \mid f(x) \neq +\infty\}$.

Let X be a given set and L be a set of functions $\ell : X \to \mathbb{R}$, which will be called the set of elementary functions. A function $f : X \to \mathbb{R}_{+\infty}$ is said to be L-convex at x^0 if $f(x^0) = \sup\{\ell(x^0) \mid \ell \in L,\ \ell \leq f\}$. Here $\ell \leq f$ means $\ell(x) \leq f(x)$ for all $x \in X$. The function $f : X \to \mathbb{R}_{+\infty}$ is called L-convex [5] if it is L-convex at any $x^0 \in X$. Given $x^0 \in \operatorname{dom} f := \{x \in X \mid f(x) < +\infty\}$, then any function $\ell \in L$, such that

$$\ell(x) - \ell(x^0) \leq f(x) - f(x^0) \text{ for all } x \in X$$

is called an L-subgradient of f at x^0. The set $\partial_L f(x^0)$ of all L-subgradients of f at x^0 is called the L-subdifferential of f at x^0. If $\partial_L f(x^0) \neq \emptyset$ then f is called L-subdifferentiable at x^0. The function f is called L-subdifferentiable, if it is L-subdifferentiable at any point $x^0 \in \operatorname{dom} f$.

It is convenient to consider sometimes the set L as a set of abstract linear functions. Then the set of the abstract affine functions H_L is defined as

$$H_L = \{h = \ell - c \mid \ell \in L,\ c \in \mathbb{R}\}.$$

Taking either L or H_L as a set of elementary functions, we can speak for L-subdifferentiability and H_L-subdifferentiability, and similarly for L-convexity and H_L-convexity. Obviously, the function $f : X \to \mathbb{R}_{+\infty}$ is H_L-subdifferentiable at $x^0 \in \operatorname{dom} f$ if and only if f is L-subdifferentiable at x^0. Recall that when L coincides with the linear functions in

$\mathbb{R}^n$, then the notions of L-subdifferentiability and H_L-convexity coincide with the usual notions of subdifferentiability and convexity from convex analysis.

Abstract convex analysis shows similarities with convex analysis, but also differences. For instance, in convex analysis, a lower semicontinuous (lsc) convex function f is subdifferentiable at a point $x^0 \in \operatorname{int}\operatorname{dom} f$ (Rockafellar [3], Theorem 24.7). A similar property in general is not true in abstract convex analysis. Taking $L = \mathcal{L}_2$ where the set $\mathcal{L}_2$ is defined in the sequel, then the function in Example 3 below is lsc and H_L-convex as underlined in Remark 2, but it is not L-subdifferentiable at the nonzero points of the coordinate axes.

A L-convex (H_L-convex) function is the upper envelope of the abstract linear (abstract affine) functions. The upper envelope of continuous functions is a lsc function, hence the lower semicontinuity is a natural assumption when dealing with abstract convex functions.

We deal in this paper with abstract subdifferentiability and abstract convexity with respect to min-type functions. Denote by $\langle \cdot, \cdot \rangle$ the inner product in $\mathbb{R}^n$, that is $\langle x, y \rangle = \sum_{j=1}^n x_j y_j$ when $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ are vectors in $\mathbb{R}^n$ (here and further the lower indices denote coordinates). For a positive integer k we define the class of abstract linear functions $\mathcal{L}_k$ (min-type functions) as the set of the functionals $\ell : \mathbb{R}^n \to \mathbb{R}$ such that $\ell(x) = \min_{1 \le i \le m} \langle l^i, x \rangle$ for some $m \le k$ and $l^1, \ldots, l^m \in \mathbb{R}^n$ (possibly repeating some of the vectors we can simply write k instead of m in the above minimum, a convention that we apply in the sequel). The set of the $\mathcal{L}_k$-affine functions $H_{\mathcal{L}_k} = \{h = \ell - c \mid \ell \in \mathcal{L}_k,\ c \in \mathbb{R}\}$ will be denoted for brevity by $\mathcal{H}_k$. Abstract convexity with respect to min-type functions is studied in [5], [6], [7], [8].

The following natural problems arise:

P_1 : *Describe the class of all $\mathcal{L}_k$-convex functions $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$.*

P_2 : *Describe the class of all $\mathcal{H}_k$-convex functions $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$.*

P_3 : *Describe the class of all $\mathcal{L}_k$-subdifferentiable functions $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$.*

Each $\mathcal{L}_k$-convex function is PH. So, only problems P_2 and P_3 are relevant to non PH functions.

Let us recall that every $\mathcal{H}_k$-convex function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is lsc and CAR ([5], Proposition 5.53). Further if $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is a lsc CAR function and $f(0) < +\infty$, then f is $\mathcal{H}_{n+1}$-convex ([5], Theorem 5.16).

We recall the notion of calmness, originating from [4], which, as seen from Theorem 1, below plays an essential role in the abstract subdifferentiability. The function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is called locally calm (of degree one) at the point $x^0 \in \operatorname{dom} f$ if

$$\operatorname{calm} f(x^0) := \liminf_{x \to x^0} \frac{f(x) - f(x^0)}{\|x - x^0\|} > -\infty\,.$$

The function f is called globally calm (of degree one) at the point $x^0 \in \operatorname{dom} f$ if

$$\operatorname{Calm} f(x^0) := \inf \left\{ \frac{f(x) - f(x^0)}{\|x - x^0\|} \mid x \in \mathbb{R}^n,\ x \ne x^0 \right\} > -\infty\,.$$

The values $\operatorname{calm} f(x^0)$ and $\operatorname{Calm} f(x^0)$ are called respectively the local and the global calmness of f at x^0. Obviously $\operatorname{Calm} f(x^0) \le \operatorname{calm} f(x^0)$, so if f is globally calm at x^0, it is also locally calm at x^0. In the sequel we deal mainly with global calmness. Here and in the sequel $\|\cdot\|$ stands for the Euclidean norm, defined by $\|x\| = \langle x, x\rangle^{1/2} = \left(\sum_{j=1}^n x_j^2\right)^{1/2}$.

The following Theorem 1 is related to problem P_3. It uses the notion of a directional derivative of a function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ at a point $x^0 \in \operatorname{dom} f$ in direction $u \in \mathbb{R}^n$ defined by $f'(x^0, u) = \lim_{t\to 0^+} \frac{1}{t}(f(x^0 + tu) - f(x^0))$. Pay attention that for a CAR function f and $x \in \operatorname{int}\operatorname{dom} f$ the directional derivatives $f'(x, \alpha x)$, $\alpha = \pm 1$, exist and are finite. Further, the notion of a local calmness $\operatorname{calm} f(x^0)$ depends only on the values of f in a neighborhood of x^0. Hence it can be introduced as above for functions being defined only in a neighborhood U of x^0, and for $x \in U$ having values in $\mathbb{R}_{+\infty}$. This remark clarifies the meaning of a local calmness used there.

Theorem 1 (Rubinov [5], Theorem 5.19) *Let f be a lsc CAR function and let $x^0 \in \operatorname{int}\operatorname{dom} f$. Suppose that either the function $x \mapsto f'(x, x)$ or the function $x \mapsto -f'(x, -x)$ is locally calm of degree one at the point x^0. Then the subdifferential $\partial_{\mathcal{L}_{n+1}} f(x^0)$ is not empty.*

As $\mathcal{L}_{k_1} \subset \mathcal{L}_{k_2}$ for $k_1 < k_2$, we see that the hypotheses of Theorem 1 implies also $\partial_{\mathcal{L}_k} f(x^0) \neq \emptyset$ for all $k \geq n+1$. Therefore of essential importance is the characterization of $\mathcal{L}_k$-subdifferentiability (and $\mathcal{H}_k$-convexity) of f when $k = n$. For this reason in the present paper we only deal with the case $k = n$.

Let $\ell \in \partial_{\mathcal{L}_n} f(x^0)$. Suppose $\ell(x) = \min_{1\leq i\leq n}\langle l^i, x\rangle$ with some $l^1, \dots, l^n \in \mathbb{R}^n$. Then $\ell(x) \leq f(x)$ for all $x \in \mathbb{R}^n$, and $\ell(x^0) = f(x^0)$. The subspace L of $\mathbb{R}^n$ given by

$$L = \{x \in \mathbb{R}^n \mid \langle l^1, x\rangle = \langle l^2, x\rangle = \cdots = \langle l^n, x\rangle\}. \tag{1}$$

is at least one-dimensional ($\dim L = 1$ when $l^1, \dots, l^n$ are linearly independent) and the restriction $f|_L$ of f on L is a convex functions (compare with [1, Theorem 5.2]). So, one could expect that, at least for a PH function f, when f is lsc CAR and the restriction $f|_L$ is convex for some one-dimensional subspace L of $\mathbb{R}^n$, then f is $\mathcal{L}_n$-subdifferentiable. However this is not the case, as shown in [1, Example 5.4]. The mentioned example reveals the difficulties that one meets, when trying to prove in a constructive way the $\mathcal{L}_n$-subdifferentiability of a concrete function. The things change, when we are looking for $\mathcal{L}_n$-subdifferentials

$$\ell(x) = \min_{1\leq i\leq n}\langle l^i, x\rangle \in \partial_{\mathcal{L}_n} f(x^0), \tag{2}$$

such that

$$\langle l^1, x^0\rangle = \langle l^2, x^0\rangle = \cdots = \langle l^n, x^0\rangle, \tag{3}$$

that is with $x^0 \in L$, where L is determined by (1). Restricting the considerations to only such $\mathcal{L}_n$-subdifferentials, we find convenient to introduce the following definitions.

Denote by $\mathcal{L}_n^0(x^0)$, $x^0 \in \mathbb{R}^n$, the set of all $\ell \in \mathcal{L}_n$ represented as in (2) for which equalities (3) are satisfied. Then obviously f is $\mathcal{L}_n^0(x^0)$-subdifferentiable at x^0 if and only if f is $\mathcal{L}_n$-subdifferentiable and $\partial_{\mathcal{L}_n} f(x^0)$ contains a $\mathcal{L}_n$-subgradient represented as in (2) for which equalities (3) are satisfied (the atypical situation from the point of view of abstract convex analysis is that the set of elementary functions varies with the variation of the point x^0). Now we say that f is $\mathcal{L}_n^0$-subdifferentiable, if f is $\mathcal{L}_n^0(x^0)$-subdifferentiable at any $x^0 \in \operatorname{dom} f$. In this case the notion of $\mathcal{L}_n^0$-subdifferentiability is not based on an underlying set of elementary functions $\mathcal{L}_n^0$. It is simply a short way to say that "f is $\mathcal{L}_n$-subdifferentiable and at each $x^0 \in \operatorname{dom} f$, f admits a $\mathcal{L}_n$-subgradient ℓ for which equalities (3) are satisfied". Introduce the abstract affine functions

$$\mathcal{H}_n^0(x^0) := H_{\mathcal{L}_n^0(x^0)} = \{h = \ell - c \mid \ \ell \in \mathcal{L}_n^0(x^0), \ c \in \mathbb{R}\}\,.$$

Following the generic definitions of abstract convex analysis we may give a sense to the assertion that "the function f is $\mathcal{L}_n^0(x^0)$-convex at x^0", or "the function f is $\mathcal{H}_n^0(x^0)$-convex at x^0". Now the notions of $\mathcal{L}_n^0$-convexity and $\mathcal{H}_n^0$-convexity can be introduced formally (and not on the basis of an underlying sets of abstract linear or abstract affine functions $\mathcal{L}_n^0$ or $\mathcal{H}_n^0$) in the following manner. We say that the function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is $\mathcal{L}_n^0$-convex ($\mathcal{H}_n^0$-convex) if f is $\mathcal{L}_n^0(x^0)$-convex ($\mathcal{H}_n^0(x^0)$-convex) at any $x^0 \in \mathbb{R}^n$.

Instead of problems P_1–P_3 we may formulate the following problems, being actually the subject of the present study. Each $\mathcal{L}_k^0$-convex function is PH. So, only problems P_2^0 and P_3^0 are relevant to non PH functions.

P_1^0 : *Describe the class of all $\mathcal{L}_k^0$-convex functions $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$.*

P_2^0 : *Describe the class of all $\mathcal{H}_k^0$-convex functions $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$.*

P_3^0 : *Describe the class of all $\mathcal{L}_k^0$-subdifferentiable functions $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$.*

Let us mention that f is globally calm at every point at which it is $\mathcal{L}_n^0$-subdifferentiable and it is CAL whenever it is $\mathcal{H}_n^0$-convex (compare with [1], Theorem 5.6). The following example shows that in general $\mathcal{L}_n^0$-subdifferentiability does not imply the CAL property.

Example 1 *Consider the function $f : \mathbb{R} \to \mathbb{R}_{+\infty}$ given by*

$$f(x) = \begin{cases} |x|\,, & |x| \le 1 \ \text{or}\ |x| \ge 2\,, \\ +\infty\,, & 1 < |x| < 2\,. \end{cases}$$

Then f is $\mathcal{L}_1^0$-subdifferentiable, but not CAL (and is not $\mathcal{H}_1^0$-convex).

3 The case of PH functions

In [1] we investigate the $\mathcal{L}_n^0$-subdifferentiability and the $\mathcal{L}_n^0$-convexity of a PH function at a given point. In this section we recall some of the obtained there results and on their basis for PH functions we solve Problem P_3^0 in Theorem 4 and Problems P_2^0 and P_3^0 in Theorem 6.

Theorem 2 ([1], Theorem 6.2) *Let $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be a CAL PH function. Then f is $\mathcal{L}_n^0$-subdifferentiable at $x^0 \in \operatorname{dom} f \setminus \{0\}$ if and only if f is globally calm at both x^0 and 0, and either $\liminf_{x \to -x^0} f(x) > -f(x^0)$ or $\operatorname{calm} f(-x^0) > -\infty$.*

The following two theorems are a straightforward consequence of the preceding theorem.

Theorem 3 *Let $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be a CAL PH function. If f is globally calm at each point $x \in \operatorname{dom} f$ then f is $\mathcal{L}_n^0$-subdifferentiable, and $\mathcal{L}_n^0$-convex.*

Theorem 4 *Let $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be a PH function. Then f is $\mathcal{L}_n^0$-subdifferentiable if and only if f is CAL, and globally calm at each point $x \in \operatorname{dom} f$.*

The following result concerns the $\mathcal{L}_n^0$-convexity at a point of PH functions.

Theorem 5 ([1], Theorem 6.3) *Let f be a lsc at zero CAL PH function. Let $x^0 \neq 0$ be a point such that f is lsc at x^0 and $-f(x^0) < \liminf_{x \to -x^0} f(x)$. Then f is $\mathcal{L}_n^0$-convex at x^0.*

As a straightforward consequence we get a characterization of the $\mathcal{L}_n^0$-convexity (and $\mathcal{H}_n^0$-convexity of a PH function.

Theorem 6 *Let $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be a PH function. Then f is $\mathcal{L}_n^0$-convex ($\mathcal{H}_n^0$-convex) if and only if f is lsc CAL and it is globally (or equivalently, locally) calm at x^0 and $-x^0$ for any $x^0 \in \operatorname{dom} f \setminus \{0\}$ such that $f(x^0) + f(-x^0) = 0$.*

From Theorem 6 we see that a PH function is $\mathcal{L}_n^0$-convex in and only if it is $\mathcal{H}_n^0$-convex. This fact can be shown also directly, slightly modifying the proof of Proposition 7.15 in [5].

4 From PH to non PH functions

The notions of $\mathcal{L}_n^0$-subdifferentiability and $\mathcal{H}_n^0$-convexity apply not only to *PH* functions. So, one would like to generalize the characterizations from Theorems 4 and 6 to non PH functions. The simplest question one can pose, is whether the obtained characterizations hold if simply the PH hypothesis is dropped without changing the remaining hypotheses and without adding new ones. The following example, related to $\mathcal{L}_n^0$-subdifferentiability shows that the thesis of Theorem 4 is not true any more (the function there is CAL and globally calm, but not $\mathcal{L}_n^0$-subdifferentiable, where $n = 2$).

Example 2 *Define the function $f : \mathbb{R}^2 \to \mathbb{R}_{+\infty}$ by*

$$f(x_1, x_2) = \begin{cases} -\sqrt{|x_1 x_2|}\,, & x_1 \geq x_2^2 \text{ and } |x_2| \geq x_1^2\,, \\ \sqrt{|x_1 x_2|}\,, & x_1 \leq -x_2^2 \text{ and } |x_2| \geq x_1^2\,, \\ 0\,, & x_1 x_2 = 0\,, \\ +\infty\,, & \text{otherwise}\,. \end{cases}$$

Then f is CAL with $f(0) < +\infty$ being globally calm at any $x \in \operatorname{dom} f$ and lsc on $\mathbb{R}^n$. At the same time f is not $\mathcal{L}_2^0$-subdifferentiable at any nonzero point of the coordinate axes (and $\mathcal{L}_2^0$-subdifferentiable at the remaining points of $\operatorname{dom} f$*). Moreover, f is not $\mathcal{L}_2$-subdifferentiable. In spite of this f is $\mathcal{H}_2^0$-convex, hence also $\mathcal{H}_2$-convex.*

Here there are some arguments about the properties that f enjoys.

1) *f is CAL.*

Consider the restriction $f^x : R^x \to \mathbb{R}_{+\infty}$, $f^x(\lambda x) = f(\lambda x)$ for $x = (x_1, x_2) \neq 0$. If $x_1 = 0$ or $x_2 = 0$ then $f^x \equiv 0$ is convex. Otherwise putting $m = \min(|x_1|, |x_2|)$ and $M = \max(|x_1|, |x_2|)$ we get the convexity of f^x from the representation

$$f^x(\lambda x) = \begin{cases} -\lambda \sqrt{|x_1 x_2|}\,, & -\frac{m}{M^2} \le \lambda \le \frac{m}{M^2}, \\ +\infty\,, & \text{otherwise.} \end{cases}$$

2) *$f(0,0) = 0$ and $f(x_1, x_2) = +\infty$ if $M = \max(|x_1|, |x_2|) > 1$.*

This property is an obvious consequence of the definition of f.

3) Calm $f(x_1^0, x_2^0) > -\infty$ *for* $x^0 = (x_1^0, x_2^0) \in \operatorname{dom} f$.

Since f is bounded from below, its global calmness is equivalent to its local calmness. At a point $x^0 \in \operatorname{dom} f \setminus \{0\}$ the function f is locally calm, since one can find a differentiable function φ defined on a neighborhood U of x^0, whose restriction of $\operatorname{dom} f \cap U$ coincides with the restriction of f on $\operatorname{dom} f \cap U$. For $x^0 = (0,\, 0)$ we have

$$\operatorname{calm} f(x^0) = \liminf_{x \to x^0} \frac{-\sqrt{|x_1 x_2|}}{\sqrt{x_1^2 + x_2^2}} = -\frac{1}{\sqrt{2}}\,.$$

4) *f is lsc on $\mathbb{R}^2$.*

This property follows from $\operatorname{dom} f$ closed and calm $f(x^0) > -\infty$ for $x^0 \in \operatorname{dom} f$.

5) *f is not $\mathcal{L}_2^0$-subdifferentiable at the nonzero points of the coordinate axes and $\mathcal{L}_2^0$-subdifferentiable at the remaining points in* $\operatorname{dom} f$.

Assume, that $\ell(x) = \min(\langle l_1, x\rangle, \langle l_2, x\rangle)$ with $l_1 = (p_1, p_2)$, $l_2 = (q_1, q_2)$ is a $\mathcal{L}_2^0$-subgradient at x^0. In the case $x^0 = (x_1^0, 0)$ with $x_1^0 \neq 0$ the equality $\langle l_1, x^0\rangle = \langle l_2, x^0\rangle = 0 = f(x^0)$ gives $p_1 = q_1 = 0$. Take the point $x^\lambda = (\lambda, \lambda^2)$ with $0 < \lambda \le 1$. The inequality

$$\ell(x^\lambda) - \ell(x^0) \le f(x^\lambda) - f(x^0) \quad \Leftrightarrow \quad \ell(x^\lambda) \le f(x^\lambda)$$

gives that for each $\lambda \in (0,\, 1]$ at least one of the inequalities $p_2\sqrt{\lambda} \le -1$, $q_2\sqrt{\lambda} \le -1$ should be satisfied. Taking $\lambda \to 0^+$ we get the contradictory inequality $0 \le -1$. The case $x^0 = (0, x_2^0)$ with $x_2^0 \neq 0$ is considered similarly. Let now $x^0 = (x_1^0, x_2^0)$ with $x_1 > 0$, $x_1 > 0$ (the existence or nonexistence of $\mathcal{L}_2^0$-subgradient at the points $(\pm x_1^0, \pm x_2^0)$ is equivalent to the same property for (x_1^0, x_2^0)). Then a $\mathcal{L}_2^0$-subgradient $\ell(x)$ exists with l_i of the type $l_i = \left(-\mu_1^i \left(x_2^0/x_1^0\right)^{1/2},\, -\mu_2^i \left(x_1^0/x_2^0\right)^{1/2}\right)$ with $\mu_1^i + \mu_2^i = 1$. We omit the rather calculative proof. Any such $\mathcal{L}_2^0$-subgradient is a $\mathcal{L}_2^0$-subgradient also at $x^0 = (0,\, 0)$.

6) *f is not $\mathcal{L}_2$-subdifferentiable.*

Every $\mathcal{L}_2^0$-subgradient is also $\mathcal{L}_2$-subgradient. Therefore f is $\mathcal{L}_2$-subdifferentiable at the points in $\operatorname{dom} f$, which are not nonzero points on the coordinate axes. For the latter f is not only non $\mathcal{L}_2^0$-subdifferentiable, but also it is not $\mathcal{L}_2$-subdifferentiable. We will prove this fact for the points $x^0 = (x_1^0, 0)$ with $x_1^0 > 0$. Assume on the contrary, that f is $\mathcal{L}_2$-subdifferentiable at such a point and let $\ell(x) = \min(\langle l_1, x\rangle, \langle l_2, x\rangle)$ be a $\mathcal{L}_2$-subgradient at x^0. Taking in mind that for arbitrary direction f is linear on every sufficiently small segment containing the origin in its relative interior, we see that ℓ should be a $\mathcal{L}_2^0$-subgradient at some point $x^1 = (x_1^1, x_2^1)$. Taking into account the symmetry of f, we may assume without loss of generality that $x_1^1 > 0$ and $x_2^1 > 0$. Now one of the forms determining $\ell(x)$, say $\langle l_1, x\rangle$ can be obtained considering the plane in $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$, the latter being the graph space of f, which passes through the origin and the points $(x_1^0, x_2^0, 0)$ and $(x_1^1, x_2^1, -\sqrt{x_1^1 x_2^1})$. We get from here $\langle l_1, x\rangle = -\sqrt{x_1^1/x_2^1}\, x_2$, where $x = (x_1, x_2)$. Moreover, it is easy to show that for x belonging to the half-plane determined by the line $x_2 = (x_2^1/x_1^1)\, x_1$ passing through the origin and the point x^1, which contains x^0, we have $\ell(x) = \langle l_1, x\rangle$. Take now the point $x^\lambda = (\lambda^2, -\lambda)$ with $0 < \lambda < 1$. Obviously $x^\lambda \in \operatorname{dom} f$ and we should have $\ell(x) = \langle l_1, x\rangle \le f(x^\lambda)$. This gives the inequality $\sqrt{x_1^1/x_2^1} \le -\sqrt{\lambda}$. Taking $\lambda \to 0^+$ we get the contradictory inequality $\sqrt{x_1^1/x_2^1} \le 0$.

7) *f is $\mathcal{H}_2^0$-convex.*

The function f is $\mathcal{H}_2^0$-convex at the points, where it is $\mathcal{L}_2^0$-subdifferentiable. According to what was said in 5) it remains to prove that f is $\mathcal{H}_2^0$-convex at the nonzero points of the coordinate axes. This is true, since if $x^0 = (x_1^0, 0)$ and $c < 0$, then the function $h : \mathbb{R}^2 \to \mathbb{R}$, $h(x) = c + \min((1/4c)x_2, -(1/4c)x_2)$ satisfies $h(x) \le f(x)$ for $x \in \mathbb{R}^2$ and $h(x^0) = f(x^0)$. The case $x^0 = (0, x_0^2)$ is treated similarly .

5 $\mathcal{L}_n^0$-subdifferentiability of non PH functions

In this section we solve problems P_3^0 for non PH functions, generalizing the results from Section 3. As Example 2 has shown, such a generalization cannot be obtained only by dropping the PH hypothesis from the results for PH functions. Some new hypotheses must be added.

Given a function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$, a point $x^0 \in \mathbb{R}^n$, a vector $u \in \mathbb{R}^n$, and a number $\zeta \in \mathbb{R}$, we introduce the function

$$\tilde{f}_{x^0,u,\zeta}(x) = \begin{cases} f(x^0) + \zeta t\,, & \text{if } x = x^0 + tu \text{ for some } t \in \mathbb{R}\,, \\ f(x)\,, & \text{otherwise}\,. \end{cases}$$

With the function f we relate the following condition:

$\mathbb{C}(f, x^0, u, \zeta)$: $$\inf_{t\in\mathbb{R}} \operatorname{Calm} \tilde{f}_{x^0,u,\zeta}(x^0 + tu) > -\infty\,.$$

or equivalently

$\mathbb{C}(f, x^0, u, \zeta)$: $$\inf_{t\in\mathbb{R}} \inf_{x\ne x^0+tu} \frac{f(x) - f(x^0) - \zeta t}{\|x - x^0 - tu\|} > -\infty\,.$$

Condition $\mathbb{C}(f, x^0, u, \zeta)$, used in Theorem 7 below, means that the graph of f admits a supporting line at $(x^0, f(x_0))$ in direction (u, ζ) along which uniform global calmness holds.

Next, as in Theorem 1, $f'(x^0, u)$ stands for the directional derivative of f at x in direction u. The following theorem is the analogue of Theorem 2 for non PH functions.

Theorem 7 *Let $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be a CAL function and let $x^0 \in \operatorname{dom} f$. Suppose that one of the following two assumptions hold:*

a) $x^0 \neq 0$ and there exists $\zeta \in \mathbb{R}$ with $-f'(x^0, -x^0) \le \zeta \le f'(x^0, x^0)$ for which condition $\mathbb{C}(f, x^0, x^0, \zeta)$ is satisfied,

b) $x^0 = 0$ and there exist $u \in \mathbb{R}^n \setminus \{0\}$ and $\zeta \in \mathbb{R}$ with $-f'(0, -u) \le \zeta \le f'(0, u)$ for which condition $\mathbb{C}(f, 0, u, \zeta)$ is satisfied.

Then f is $\mathcal{L}_n^0$-subdifferentiable at x^0.

Proof a) Let $x^0 \neq 0$. Due to condition $\mathbb{C}(f, x^0, x^0, \zeta)$ there exists a constant $C > 0$ such that

$$\inf_{t \in \mathbb{R}} \operatorname{Calm} \tilde{f}_{x^0, x^0, \zeta}(x^0 + t x^0) \ge -C > -\infty \,. \tag{4}$$

Consider the subspace $M = \{x \in \mathbb{R}^n \mid \langle x^0, x \rangle = 0\}$ of $\mathbb{R}^n$ orthogonal to the vector x^0. Let $\varepsilon > 0$. Since M is an $(n-1)$-dimensional space, we can find n vectors $m^1, \dots, m^n \in M$ such that their convex hull S, which is a simplex, contains the ball $B_\varepsilon = \{x \in M \mid \|x\| \le \varepsilon\}$. Let $q(x) = \max_{1 \le i \le n} \langle m^i, x \rangle$ be the support function of S. Since $S \supset B_\varepsilon$ and the support function of B_ε is equal to $\varepsilon \|x\|$ for $x \in M$, it follows that

$$q(x) = \max_{1 \le i \le n} \langle m^i, x \rangle \ge \varepsilon \|x\|, \quad x \in M \,. \tag{5}$$

Fix $x \in \mathbb{R}^n$ and let $\bar{x} = \dfrac{\langle x^0, x \rangle}{\|x^0\|^2} \, x^0$ be the orthogonal projection of x on $L := \{\lambda x^0 \mid \lambda \in \mathbb{R}\}$. Since $\bar{x} = x^0 + \left(\dfrac{\langle x^0, x \rangle}{\|x^0\|^2} - 1 \right) x^0$, we have

$$\tilde{f}_{x^0, x^0, \zeta}(\bar{x}) = f(x^0) + \zeta \left(\frac{\langle x^0, x \rangle}{\|x^0\|^2} - 1 \right) . \tag{6}$$

Since $\bar{x} \in L$, from (4) ge have

$$\tilde{f}_{x^0, x^0, \zeta}(x) - \tilde{f}_{x^0, x^0, \zeta}(\bar{x}) \ge -C \, \|x - \bar{x}\| \,.$$

Due to (5) we get

$$\|x - \bar{x}\| \le \frac{1}{\varepsilon} \max_{1 \le i \le n} \langle m^i, x - \bar{x} \rangle \,,$$

so that

$$\tilde{f}_{x^0, x^0, \zeta}(x) - \tilde{f}_{x^0, x^0, \zeta}(\bar{x}) \ge -C \, \|x - \bar{x}\| \ge -\frac{C}{\varepsilon} \max_{1 \le i \le n} \langle m^i, x - \bar{x} \rangle \,.$$

Since $m^i \in M$, $i = 1, \dots, n$, and $\bar{x}$ belongs to the subspace L being orthogonal to M, it follows that $\langle m^i, \bar{x} \rangle = 0$ for $i = 1, \dots, n$. Using these equalities and (6) we obtain

$$\begin{aligned} f(x) \ge \tilde{f}_{x^0, x^0, \zeta}(x) &= \left(\tilde{f}_{x^0, x^0, \zeta}(x) - \tilde{f}_{x^0, x^0, \zeta}(\bar{x}) \right) + \tilde{f}_{x^0, x^0 \zeta}(\bar{x}) \\ &\ge -\frac{C}{\varepsilon} \max_{1 \le i \le n} \langle m^i, x \rangle + f(x^0) + \zeta \left(\frac{\langle x^0, x \rangle}{\|x^0\|^2} - 1 \right) , \end{aligned}$$

or equivalently

$$f(x) - f(x^0) \geq \min_{1\leq i\leq n} \left\langle -\frac{C}{\varepsilon}\, m^i + \frac{\zeta}{\|x^0\|^2}\, x^0,\, x \right\rangle - \zeta\,. \tag{7}$$

Here we have used the inequality $f(x) \geq \tilde{f}_{x^0,x^0,\zeta}(x)$ which needs to be explained only when $x = x^0 + tx^0$. Then this inequality reduces to

$$f(x^0 + tx^0) - f(x^0) \geq \zeta t\,,$$

which follows from the convexity of the function $t \mapsto f(tx^0)$. Indeed, for $t = 0$ the inequality turns into an obvious equality. For $t > 0$ it follows from

$$\frac{f(x^0 + tx^0) - f(x^0)}{t} \geq f'(x^0, x^0) \geq \zeta\,.$$

For $t < 0$ it follows from

$$\frac{f(x^0 - t(-x^0)) - f(x^0)}{-t} \geq f'(x^0, -x^0) \geq -\zeta\,.$$

Put now

$$l^i = -\frac{C}{\varepsilon}\, m^i + \frac{\zeta}{\|x^0\|^2}\, x^0\,, \quad i = 1, \dots, n\,,$$

and observe that these vectors do not depend on x (from here on x could be considered an arbitrary vector). Define the functional $\ell : \mathbb{R}^n \to \mathbb{R}$ by $\ell(x) = \min_{1\leq i\leq n}\langle l^i, x\rangle$. We have obviously $\langle l^i, x^0\rangle = \zeta$ for $i = 1, \dots, n$, whence $\ell \in \mathcal{L}_n^0(x^0)$ and $\ell(x^0) = \zeta$. Now inequality (7) can be written as

$$f(x) - f(x^0) \geq \ell(x) - \ell(x^0)\,,$$

which shows that $\ell \in \partial_{\mathcal{L}_n^0(x^0)}$, that is f is $\mathcal{L}_n^0$-subdifferentiable at x^0.

b) Let $x^0 = 0$. Define the function $g : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ by

$$g(x) = \begin{cases} f(0) + \zeta t\,, & x = tu\,,\\ f(x)\,, & \text{otherwise}\,. \end{cases}$$

Obviously g is CAL, and affine along the line $R^u = \{tu \mid t \in \mathbb{R}\}$. The latter shows that $-g'(u, -u) = \zeta = g'(u, u)$. We will use the inequality $f(x) \geq g(x)$ which should be explained only for $x = tu$. Then it turns into $f(tu) \geq f(0) + \zeta t$. For $t = 0$ it is obviously true. For $t > 0$ it follow from $(f(tu) - f(0))/t \geq f'(0, u) \geq \zeta$. For $t < 0$ it follows from $(f(-t(-u)) - f(0))/(-t) \geq f'(0, -u) \geq -\zeta$.

It is easy to verify that $\tilde{g}_{u,u,\zeta} = \tilde{f}_{0,u,\zeta}$, whence conditions $\mathbb{C}(f, 0, u, \zeta)$ and $\mathbb{C}(g, u, u, \zeta)$ are equivalent. Since $\mathbb{C}(f, 0, u, \zeta)$ is true, also $\mathbb{C}(g, u, u, \zeta)$ holds. Since $u \neq 0$, on the base of a) there exists $\ell \in \partial_{\mathcal{L}_n^0} g(u)$. Let $\ell(x) = \min_{1\leq i\leq n}\langle l^i, x\rangle$ with $\langle l^i, u\rangle = \ell(u) = \zeta$, $i = 1, \dots, n$. The equality $\ell(u) = \zeta$ follows from the linearity of ℓ on the line R^u and the inequality

$$g(tu) - g(u) \geq \ell(tu) - \ell(u) \quad \Leftrightarrow \quad (t-1)\zeta \geq (t-1)\ell(u)\,, \quad \forall t \in \mathbb{R}\,.$$

(this gives $\zeta \geq \ell(u)$ for $t > 1$, and $\zeta \leq \ell(u)$ for $t < 1$). Recalling that to $\ell(0) = 0 = \langle l^i, 0\rangle$, $i = 1, \dots, n$, we have

$$f(x) \geq g(x) \geq g(u) + \ell(x) - \ell(u) = f(0) + \zeta + \ell(x) - \zeta = f(0) + \ell(x) - \ell(0)\,,$$

which shows that f is $\mathcal{L}_n^0$-subdifferentiable at $x^0 = 0$. ■

Theorem 2 is a straightforward corollary of Theorem 7. Indeed, if the PH function f satisfies the hypotheses of Theorem 2, then according to Proposition 4.5 in [1], it satisfies also condition $\mathbb{C}(f, x^0, x^0, \zeta)$ with $\zeta = f'(x^0, x^0) = f'(0, x^0)$. Thus, the hypotheses of Theorem 7 a) are satisfied, whence f is is $\mathcal{L}_n^0$-subdifferentiable at x^0.

The following theorem gives a characterization of the $\mathcal{L}_n^0$-subdifferentiability of a non PH function and is an analogue of Theorem 4.

Theorem 8 *Let the function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be $\mathcal{H}_n^0$-convex at the points $x \notin \operatorname{dom} f$. Then f is $\mathcal{L}_n^0$-subdifferentiable if and only if it is CAL, satisfies condition $\mathbb{C}(f, x^0, x^0, \zeta)$ with some $\zeta \in [-f'(x^0, -x^0),\, f'(x^0, x^0)]$ at any $x^0 \in \operatorname{dom} f \setminus \{0\}$, and when $0 \in \operatorname{dom} f$ it satisfied condition $\mathbb{C}(f, 0, u, \zeta)$ with some $u \in \mathbb{R}^n \setminus \{0\}$ and some $\zeta \in [-f'(0, -u),\, f'(0, u)]$.*

Proof *Necessity.* Let f be $\mathcal{L}_n^0$-subdifferentiable. Then f is CAL according to Theorem 5.6 in [1].

Let $x^0 \in \operatorname{dom} f \setminus \{0\}$ and $\ell \in \partial_{\mathcal{L}_n^0} f(x^0)$. Suppose that $\ell(x) = \min_{1 \leq i \leq n} \langle l^i, x\rangle$ with $\langle l^i, x^0\rangle = \ell(x^0)$ for $i = 1, \dots n$. Put $\zeta = \ell(x^0)$. We have

$$f(x) - f(x^0) \geq \ell(x) - \ell(x^0)\,, \quad \forall x \in \mathbb{R}^n\,.$$

When $x = x^0 + tx^0$ this gives

$$f(x^0 + tx^0) - f(x^0) \geq t\ell(x^0) = t\zeta\,, \quad \forall t \in \mathbb{R}\,,$$

which implies easily $-f'(x^0, -x^0) \leq \zeta \leq f'(x^0, x^0)$. Now

$$\begin{aligned}
\operatorname{Calm} \tilde{f}_{x^0,x^0,\zeta}(x^0 + tx^0) &= \inf_{\substack{x \in \mathbb{R}^n \\ x \neq x^0 + tx^0}} \frac{\tilde{f}_{x^0,x^0,\zeta}(x) - \tilde{f}_{x^0,x^0,\zeta}(x^0 + tx^0)}{\|x - x^0 - tx^0\|} \\
&\geq \min\left(\inf_{\substack{x \in \mathbb{R}^n \\ x \neq x^0 + tx^0}} \frac{f(x) - f(x^0) - t\ell(x^0)}{\|x - x^0 - tx^0\|}\,,\ \inf_{\substack{x = x^0 + sx^0 \\ s \neq t}} \frac{\tilde{f}_{x^0,x^0,\zeta}(x) - f(x^0) - t\zeta}{\|x - x^0 - tx^0\|} \right) \\
&\geq \min\left(\inf_{\substack{x \in \mathbb{R}^n \\ x \neq x^0 + tx^0}} \frac{\ell(x) - \ell(x^0) - t\ell(x^0)}{\|x - x^0 - tx^0\|}\,,\ \inf_{\substack{s \in \mathbb{R} \\ s \neq t}} \frac{(s - t)\,\zeta}{|s - t|\,\|x^0\|} \right) \\
&\geq \min\left(-\min_{1 \leq i \leq n} \|l^i\|\,,\ -\frac{|\zeta|}{\|x^0\|} \right).
\end{aligned}$$

The right hand side of this inequality is finite and does not depend on t, whence condition $\mathbb{C}(f, x^0, x^0, \zeta)$ is satisfied.

Let $0 \in \operatorname{dom} f$ and $\ell \in \mathcal{L}_n^0(0)$. Suppose that $\ell(x) = \min_{1 \leq i \leq n} \langle l^i, x\rangle$. The system (of $n - 1$) linear homogeneous equations (in n variables)

$$\langle l^1, x\rangle = \langle l^2, x\rangle = \cdots = \langle l^n, x\rangle$$

has rank at most $n-1$. Therefore it possesses a solution $u \neq 0$. Put $\zeta = \ell(u)$. We have $f(x) - f(0) \geq \ell(x)$, $\forall x \in \mathbb{R}^n$. When $x = tu$ this gives $f(tu) - f(0) \geq t\ell(u) = t\zeta$, $\forall t \in \mathbb{R}$, which implies easily $-f'(0,-u) \leq \zeta \leq f'(0,u)$. Now

$$\begin{aligned}
\operatorname{Calm} \tilde f_{0,u,\zeta}(tu) &= \inf_{\substack{x\in\mathbb{R}^n\\ x\neq tu}} \frac{\tilde f_{0,u,\zeta}(x) - \tilde f_{0,u,\zeta}(tu)}{\|x - tu\|} \\
\geq \min &\left(\inf_{\substack{x\in\mathbb{R}^n\\ x\neq tu}} \frac{f(x) - f(0) - t\ell(u)}{\|x-tu\|} \, , \, \inf_{\substack{x=su\\ s\neq t}} \frac{\tilde f_{0,u,\zeta}(x) - f(0) - t\zeta}{\|x - tu\|} \right) \\
&\geq \min \left(\inf_{\substack{x\in\mathbb{R}^n\\ x\neq tu}} \frac{\ell(x) - t\ell(u)}{\|x-tu\|} \, , \, \inf_{\substack{s\in\mathbb{R}\\ s\neq t}} \frac{(s-t)\,\zeta}{|s-t|\,\|u\|} \right) \\
&\geq \min \left(- \min_{1\leq i\leq n} \|l^i\| \, , \, -\frac{|\zeta|}{\|u\|} \right) .
\end{aligned}$$

The right hand side of this inequality is finite and does not depend on t, whence condition $\mathbb{C}(f,0,u,\zeta)$ is satisfied.

Sufficiency. The sufficiency is established by Theorem 7. ■

Remark 1 *In Theorem 8 the hypothesis that f is $\mathcal{H}_n^0$-convex at the points $x \notin \operatorname{dom} f$ is used only in the necessity to obtain that f is CAL.*

Now we give an explanation for the lack $\mathcal{L}_2^0$-differentiability of the function in Example 2 in the framework of Theorems 7 and 8. Let x^0 be a non-zero point on the coordinate axes. We confine to the case $x^0 = (x_1^0,\, 0)$, $x_1^0 > 0$ (the other cases are similar). It is clear that $f'(x^0,x^0) = f'(x^0,-x^0) = 0$. Therefore $\zeta \in [-f'(x^0,-x^0),\, f'(x^0,x^0)]$ implies $\zeta = 0$. Now it is clear that $\tilde f_{x^0,x^0,0} = f$. The points $x^\lambda = (\lambda, 0)$ with $0 < \lambda \leq 1$ belong to the line $\{x^0 + tx^0 \mid t \in \mathbb{R}\}$. For any such point we have

$$\operatorname{Calm} f(x^\lambda) \leq \frac{f(\lambda,\lambda^2) - f(x^\lambda)}{\|(\lambda,\lambda^2) - x^\lambda\|} = -\frac{\sqrt{\lambda^3}}{\lambda^2} = -\frac{1}{\sqrt{\lambda}} \, .$$

Though f is globally calm at each point in $\operatorname{dom} f$, we have

$$\inf_{t\in\mathbb{R}} \tilde f_{x^0,x^0,0}(x^0 + tx^0) \leq \inf_{0<\lambda\leq 1} -\frac{1}{\sqrt{\lambda}} = -\infty \, .$$

Therefore condition $\mathbb{C}(f,x^0,x^0,\zeta)$ is not satisfied and consequently from the Necessity of Theorem 8 the function f is not $\mathcal{L}_2^0$-subdifferentiable at x^0.

6 $\mathcal{H}_n^0$-convexity for non PH functions

Let $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$, $\bar x \in \mathbb{R}^n$, $c \in \mathbb{R}$, and $v \in \mathbb{R}^n$. We define the function $f^{\bar x,c,v} : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ by

$$f^{\bar x,c,v}(x) = \begin{cases} \operatorname{co}\,\left(f|_{\bar x+tv,\, t\in\mathbb{R}},\ \chi_{x^0}|_{\bar x+tv,\, t\in\mathbb{R}} + c\right)(x)\, , & x = tv,\ ,t\in\mathbb{R}\, , \\ f(x)\, , & \text{otherwise}\, . \end{cases}$$

Here χ_{x^0} stands for the indicator function

$$\chi_{x^0}(x) = \begin{cases} 0\,, & x = x^0\,, \\ +\infty\,, & x \in \mathbb{R}^n \setminus \{x^0\}\,, \end{cases}$$

$g|_{\bar{x}+tv,\, t\in\mathbb{R}}$ denotes the restriction of a function g on the line $\{\bar{x}+tv \mid t \in \mathbb{R}\}$, and $\mathrm{co}\,(g_1, g_2)$ is the convex hull of the functions g_1, g_2.

Obviously, when f is CAL, and $\bar{x} = su$ for some $s \in \mathbb{R}$, then $f^{\bar{x},c,u}$ is CAL. Further, the condition that f is $\mathcal{H}_n^0$-convex at $x^0 \in \mathbb{R}^n$ is equivalent to the condition that the function

$$g(x) = \begin{cases} c\,, & x = x^0\,, \\ f(x)\,, & x \in \mathbb{R}^n \setminus \{x^0\}\,, \end{cases}$$

is $\mathcal{L}_n^0$-subdifferentiable for any $c < f(x^0)$ (hence the same is to be said for $f^{x^0,x^0,c}$) when $x^0 \neq 0$, or for $f^{0,u,c}$ with suitable $u \neq 0$ when $x^0 = 0$. This observation and Theorems 7 and 8 give immediately the following results:

Theorem 9 *Let $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be a CAL function and let $x^0 \in \mathbb{R}^n$. Suppose that for any $c < f(x^0)$ one of the following two assumptions hold:*

a) $x^0 \neq 0$ and there exists $\zeta \in \mathbb{R}$ with

$$-(f^{x^0,c,x^0})'(x^0, -x^0) \leq \zeta \leq (f^{x^0,c,x^0})'(x^0, x^0)$$

for which condition $\mathbb{C}(f^{x^0,c,x^0}, x^0, x^0, \zeta)$ is satisfied,

b) $x^0 = 0$ and there exist $u \in \mathbb{R}^n \setminus \{0\}$ and $\zeta \in \mathbb{R}$ with

$$-(f^{0,c,u})'(0, -u) \leq \zeta \leq (f^{0,c,u})'(0, u)$$

for which condition $\mathbb{C}(f^{0,c,u}, 0, u, \zeta)$ is satisfied.

Then f is $\mathcal{H}_n^0$-convex at x^0.

Theorem 10 *The function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is $\mathcal{H}_n^0$-convex if and only if it is CAL, and for any $x^0 \in \mathbb{R}^n$ and any $c < f(x^0)$:*

a) when $x^0 \neq 0$ it holds $\mathbb{C}(f^{x^0,c,x^0}, x^0, x^0, \zeta)$ with some

$$\zeta \in [-(f^{x^0,c,x^0})'(x^0, -x^0),\ (f^{x^0,c,x^0})'(x^0, x^0)]\,,$$

b) when $x^0 = 0$ it holds $\mathbb{C}(f^{0,c,u}, 0, u, \zeta)$ with some $u \in \mathbb{R}^n \setminus \{0\}$ and some

$$\zeta \in [-(f^{0,c,u})'(0, -u),\ (f^{0,c,u})'(0, u)]\,.$$

Theorem 10 characterizes the $\mathcal{H}_n^0$-convexity and solves Problem P_2^0. However, one can expect a more simple and effective characterization. As in convex analysis, one expects a characterization of (abstract) convexity (in sense of upper envelopes of elementary functions) rather in terms of lower semicontinuity. Due to this remark, the following natural question arises:

Is a lsc CAL function $\mathcal{H}_n^0$-convex?

At first glance Propositions 5.53 and Theorem 5.16 in [5] support a positive answer. However, in spite of this, the answer of this question is negative, as seen from the following example:

Example 3 *The function* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$f(x_1, x_2) = \begin{cases} -\sqrt{|x_1 x_2|}\,, & x_1 \geq 0\,, \\ \sqrt{|x_1 x_2|}\,, & x_1 < 0\,, \end{cases}$$

is lsc (moreover, it is continuous), its restriction on any line $R^x = \{tx \mid t \in \mathbb{R}\}$ *is linear (hence* f *is PH CAL), but* f *is not* $\mathcal{H}_2^0$*-convex at the points* x^0 *belonging to the coordinate axes.*

We give an explanation of example 3 for the case $x^0 = (0,\, x_2^0)$, $x_2^0 > 0$. The case of any other point on the coordinate axes is explained similarly. Let $h = \ell - c \in \mathcal{H}_2^0(x^0)$, $h \leq f$, when $\ell \in \mathcal{L}_2^0(x^0)$, $c \in \mathbb{R}$. Since f is PH, we have

$$h(tx) = \ell(tx) - c = t\ell(x) - c \leq f(tx) = tf(x)\,.$$

Dividing by $t > 0$ and letting $t \to +\infty$ we get $\ell(x) \leq f(x)$, $\forall x \in \mathbb{R}^n$. Since ℓ is linear on $R^{x^0} = \{tx^0 \mid t \in \mathbb{R}\}$, we have

$$0 = \ell(-x^0) + \ell(x^0) \leq f(-x^0) + f(x^0) = 0\,,$$

whence $\ell(x^0) = f(x^0) = 0$. This shows that f is $\mathcal{L}_2^0$-subdifferentiable at x^0. According to Theorem 2 (or Theorem 7 with account that $\tilde{f}_{x^0, x^0, f(x^0)/\|x^0\|} = f$) we should have $\operatorname{calm} f(x^0) > -\infty$. At the same time

$$\operatorname{calm} f(x^0) \leq \liminf_{x_1 \to 0^+} \frac{f(x_1, x_2^0) - f(0, x_2^0)}{\|(x_1, x_2^0) - (0, x_2^0)\|} = \liminf_{x_1 \to 0^+} \frac{\sqrt{x_1 x_2^0}}{x_1} = -\infty\,.$$

Example 3 shows that the property of f being lsc CAL, without some additional condition, is not enough to guarantee the $\mathcal{H}_n^0$-convexity of f, even when f is PH.

Confining to PH functions, we have the following result:

Theorem 11 *Let* $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ *be PH function. Then* f *is* $\mathcal{H}_n^0$*-convex (*$\mathcal{L}_n^0$*-convex) if and only if* f *is lsc CAL, and is globally calm at any* x^0 *such that* $x^0 \in \operatorname{dom} f \setminus \{0\}$ *and* $f(-x^0) = -f(x^0)$.

Proof *Necessity.* If f is $\mathcal{H}_n^0$-convex, then f is lsc CAR according to Proposition 5.53 in [5]. Since f is PH, it is $\mathcal{H}_n^0$-convex if and only if it is $\mathcal{L}_n^0$-convex. Now from Theorem 2 we get that f is globally calm at x^0.

Sufficiency. Let f be lsc CAL. From Proposition 4.4 in [1] we have $\operatorname{Calm} f(0) > -\infty$. Assume that f is calm at any $x^0 \in \operatorname{dom} f$ such that $f(-x^0) + f(x^0) = 0$. Then the function f is globally calm at x^0 and at $-x^0$ (the global calmness at $-x_0$ follows from the hypotheses observing that $f(-(-x^0)) + f(-x^0) = 0$). Now from Theorem 2 we have that f is $\mathcal{L}_n^0$-subdifferentiable at x^0 (and at 0), hence also $\mathcal{L}_n^0$-convex and $\mathcal{H}_n^0$-convex at x^0 (and at 0).

Let now $x^0 \in \mathbb{R}^n \setminus \{0\}$ and $f(-x^0) + f(x^0) > 0$ (the CAL property of f excludes the case $f(-x^0)+f(x^0) < 0$). Take $c \in \mathbb{R}$ such that $c < f(x^0)$ and still $f(-x^0)+c > 0$. Define the function $g : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ by

$$g(x) = \begin{cases} tc\,, & x = tx^0,\ t \geq 0\,, \\ f(x)\,, & \text{otherwise}\,. \end{cases}$$

Obviously g is lsc PH and CAL. Since f is lsc, in particular it is globally calm at 0. Therefore there exists $C > 0$ such that $\operatorname{Calm} g(0) > -C$. We show that g is globally calm at x^0. Since f is lsc at x^0, there exists $\delta > 0$ such that $c < f(x)$ for $\|x - x^0\| \leq \delta$. Then for $\|x - x^0\| \leq \delta$ it is easy to derive

$$g(x) - g(x^0) = -\frac{|c|}{\|x^0\|}\,\|x - x^0\|\,.$$

Let $C_1 > C$. Then there exists $m > 0$ such that $g(x) - g(x^0) \geq -C_1\|x - x^0\|$ for $\|x - x^0\| \geq m$. Otherwise we would have a sequence x^k with $\|x^k\| \to \infty$ such that $g(x^k) - g(x^0) < -C_1\,\|x^k - x^0\|$ which gives

$$-C \leq \frac{g(x^k)}{\|x^k\|} < \frac{g(x^0)}{\|x^k\|} - C_1\,\frac{\|x^k - x^0\|}{\|x^k\|}$$

with the contradictory inequality $-C \leq -C_1$ when passing to the limit. Since g is lsc, $\inf_{\delta \leq \|x - x^0\| \leq m}(g(x) - g(x^0)) > -C_2 > -\infty$ for some $C_2 > 0$. Therefore for such x we have

$$g(x) - g(x^0) \geq -C_2 = -\,\frac{C_2}{\delta}\,\|x - x^0\|\,.$$

Unifying the three cases, we have for all $x \in \mathbb{R}^n \setminus \{x^0\}$ the inequality

$$\frac{g(x) - g(x^0)}{\|x - x^0\|} \geq -\,\max\left(\frac{|c|}{\|x^0\|}\,,\, C_1,\, \frac{C_2}{\delta}\,,\right)$$

which shows that g is globally calm at x^0. Since $g(-x^0) + g(x^0) > 0$, Theorem 2 shows that there exists $\ell \in \partial_{\mathcal{L}_n^0} g(x^0)$. From $g \leq f$ we have $\ell \leq f$. Further we have $\ell(x^0) = c$ (and $\ell(0) = 0$) and since $c < f(x^0)$ can be chosen arbitrary close to $f(x^0)$, we get that f is $\mathcal{L}_n^0$-convex at x^0. We have actually shown that such an assertion takes place for arbitrary $x^0 \in \mathbb{R}^n$, which proves that f is $\mathcal{L}_n^0$-convex, whence also $\mathcal{H}_n^0$-convex. ■

In the framework of Theorem 11 for Example 3 we can explain the lack of $\mathcal{H}_2^0$- convexity at the (nonzero) points x^0 of the coordinate axes. Since at such a point we have $f(-x^0) + f(x^0) = 0$, at these points besides the lower semicontinuity, we must have the global calmness property. The latter however does not hold.

Remark 2 *Let us observe, that in Example 9 though at the points of the coordinate axes the function f is not $\mathcal{H}_2^0$-convex, at these points it is still $\mathcal{H}_2$-convex (and $\mathcal{L}_2$-convex).*

Let us e.g. demonstrate the $\mathcal{L}_2$-convexity (and hence the $\mathcal{H}_2$-convexity) of f at the point $x^0 = (1,0)$. For $s > 0$ define the functionals $\ell_s \in \mathcal{L}_2$ by

$$\ell_s(x_1,x_2) = \min\left(-\frac{1}{\sqrt{s}}\,x_2,\ -\frac{3}{2}\,\sqrt{s}\,x_1 + \frac{1}{2\sqrt{s}}\,x_2\right)\,,$$

which can be written also as

$$\ell_s(x_1,x_2) = \begin{cases} -\dfrac{1}{\sqrt{s}}\,x_2, & x_2 \geq s x_1, \\ -\dfrac{3}{2}\,\sqrt{s}\,x_1 + \dfrac{1}{2\sqrt{s}}\,x_2, & x_2 < s x_1\,. \end{cases}$$

It holds $\ell_s \leq f$, which follows from the cases:

1^0. $x_1 \geq 0$, $x_2 \geq 0$, $x_2 \geq s x_1$:

$$\ell_s(x_1,x_2) = -\frac{1}{\sqrt{s}}\,x_2 = -\frac{1}{\sqrt{s}}\,\sqrt{x_2\,x_2} \leq -\frac{1}{\sqrt{s}}\,\sqrt{s x_1\,x_2} = -\sqrt{x_1\,x_2} = f(x_1,x_2)\,.$$

2^0. $x_1 < 0$, $x_2 \geq 0$, $x_2 \geq s x_1$:

$$\ell_s(x_1,x_2) = -\frac{1}{\sqrt{s}}\,x_2 \leq \sqrt{-x_1\,x_2} = f(x_1,x_2)\,.$$

3^0. $x_1 < 0$, $x_2 < 0$, $x_2 \geq s x_1$:

$$\ell_s(x_1,x_2) = -\frac{1}{\sqrt{s}}\,x_2 = \frac{1}{\sqrt{s}}\,\sqrt{(-x_2)\,(-x_2)} \leq \frac{1}{\sqrt{s}}\,\sqrt{(-s x_1)\,(-x_2)} = \sqrt{x_1\,x_2} = f(x_1,x_2)\,.$$

4^0. $x_1 \geq 0$, $x_2 \geq 0$, $x_2 < s x_1$:

$$\begin{aligned}\ell_s(x_1,x_2) &= -\frac{3}{2}\,\sqrt{s}\,x_1 + \frac{1}{2\sqrt{s}}\,x_2 = -\frac{3}{2}\,\sqrt{x_1\,s x_1} + \frac{1}{2\sqrt{s}}\,\sqrt{x_2\,x_2} \\ &< -\frac{3}{2}\,\sqrt{x_1 x_2} + \frac{1}{2\sqrt{s}}\,\sqrt{s x_1\,x_2} = -\sqrt{x_1\,x_2} = f(x_1,x_2)\,.\end{aligned}$$

5^0. $x_1 \geq 0$, $x_2 < 0$, $x_2 < s x_1$:

$$\ell_s(x_1,x_2) = -\frac{3}{2}\,\sqrt{s}\,x_1 + \frac{1}{2\sqrt{s}}\,x_2 \leq -\frac{1}{2}\left(\sqrt{s}\,x_1 + \frac{1}{\sqrt{s}}\,(-x_2)\right) \leq \sqrt{-x_1\,x_2} = f(x_1,x_2)\,.$$

6^0. $x_1 < 0$, $x_2 < 0$, $x_2 < s x_1$:

$$\ell_s(x_1,x_2) = -\frac{3}{2}\,\sqrt{s}\,x_1 + \frac{1}{2\sqrt{s}}\,x_2 = \frac{3}{2}\,\sqrt{(-x_1)\,(-s x_1)} - \frac{1}{2\sqrt{s}}\,\sqrt{(-x_2)\,(-x_2)}$$

$$< \frac{3}{2}\sqrt{(-x_1)(-x_2)} - \frac{1}{2\sqrt{s}}\sqrt{(-sx_1)(-x_2)} = \sqrt{x_1 x_2} = f(x_1, x_2)\,.$$

Taking into account $\ell_s(x^0) = \ell_s(1,0) = -\frac{3}{2}\sqrt{s} \to 0 = f(x^0)$ as $s \to 0^+$, we get the $\mathcal{L}_2$-convexity (and the $\mathcal{H}_2$-convexity) of f at x^0.

Though we arrive at the notion of $\mathcal{H}_n^0$-convexity wishing to study the $\mathcal{H}_n$-convexity, the above example shows that the two notions are not equivalent even for lsc PH CAL functions. Obviously, $\mathcal{H}_2^0$-convexity implies $\mathcal{H}_2$-convexity, but the converse is not true.

Theorem 11 shows, that a lsc CAL function need not be $\mathcal{H}_n^0$-convex. In the case of a PH function we have to add as a hypothesis the global calmness at some special points. One can expect that for non PH functions a similar result holds. Actually, the following theorem has place.

Theorem 12 *Let the function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be $\mathcal{H}_n^0$-convex. Then f is lsc CAL, and for any one-dimensional subspace $L \subset \mathbb{R}^n$ such that the restriction $f|_L$ is an affine function and any $\varepsilon > 0$ it holds*

$$\inf_{u\in L}\ \inf_{x\in\mathbb{R}^n\setminus\{u\}} \frac{f(x) - f(u) + \varepsilon}{\|x-u\|} > -\infty,\,. \tag{8}$$

Proof Let f be $\mathcal{H}_n^0$-convex. Then f is lsc at any $x^0 \in \mathbb{R}^n$. Let L be one-dimensional subspace of $\mathbb{R}^n$. Fix x^-, $x^+ \in L$. Let $x^0 = (1-\tau)x^- + \tau x^+$ with $0 < \tau < 1$. Consider the cases:

a) $x^0 \neq 0$. Let $c < f(x^0)$. Since f is $\mathcal{H}_n^0$-convex at x^0, for some $\ell(x) = \min_{1\le i\le n}\langle l^i, x\rangle$ satisfying (3) and some $\bar{c} \in \mathbb{R}$ the function $h(x) = \ell(x) - \bar{c}$ satisfies $h(x) < f(x)$ for all $x \in \mathbb{R}^n$ and $c < h(x^0)$. With regard that the restriction $f|_L$ is affine we get

$$(1-\tau)f(x^-) + \tau f(x^+) \ge (1-\tau)h(x^-) + \tau h(x^+) = h(x^0) > c\,.$$

Since $c < f(x^0)$ is arbitrary, we get

$$(1-\tau)f(x^-) + \tau f(x^+) \ge f(x^0)\,. \tag{9}$$

b) $x^0 = 0$. Let $x^k \to x^0$ with $x^k \in L \setminus \{0\}$. Put $x^k = (1-\tau_k)x^- + \tau_k x^+$. Obviously $\tau_k \to \tau$. From the proved lower semicontinuity and the proved inequality (9) in a) we get

$$f(x^0) \le \liminf_k f(x^k) \le \liminf_k \big((1-\tau_k)f(x^-) + \tau_k f(x^+)\big) = (1-\tau)f(x^-) + \tau f(x^+)\,.$$

Thus, inequality (9) is true for all $\tau \in (0,\,1)$ which shows that f is CAL.

Let $f|_L$ be affine on the space $L = \{tx^0 \mid t \in \mathbb{R}\}$ where $x^0 \in \mathbb{R}^n \setminus \{0\}$. Put $c = f(x^0) - \varepsilon$ where $\varepsilon > 0$. The function f^{x^0,c,x^0} is given by

$$f^{x^0,c,x^0}(x) = \begin{cases} f(x) - \varepsilon\,, & x \in L\,,\\ f(x)\,, & \text{otherwise}\,. \end{cases}$$

Put for brevity $g = f^{x^0,c,x^0}$. Now $g'(x^0, x^0) = -g'(x^0, -x^0) = f(x^0) - f(0)$ because $g|_L = f|_L - \varepsilon$ is an affine function. Therefore $\zeta \in [-g'(x^0,-x^0),\, g'(x^0,x^0)]$ means $\zeta =$

$f(x^0) - f(0)$. With this choice of ζ we have $\tilde{g}_{x^0,x^0,\zeta}(x) = g(x)$. It is clear now that (8) is a reformulation of condition $\mathbb{C}(g, x^0, x^0, \zeta)$, true on the base of Theorem 10. ■

The next Theorem is a reversal of Theorem 12 (an open question is whether the condition $0 \in \operatorname{dom} f$ which lacks in Theorem 12 can be removed from the hypotheses of Theorem 13).

Theorem 13 *Let the function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ with $0 \in \operatorname{dom} f$ be lsc and CAL. Suppose that for any 1-dimensional subspace $L \subset \mathbb{R}^n$ such that the restriction $f|_L$ is an affine function and any $\varepsilon > 0$ condition (8) holds. Then f is $\mathcal{H}_n^0$-convex.*

Proof Initially our considerations will not use the assumption $0 \in \operatorname{dom} f$.

Suppose that $f|_L$ is affine for some 1-dimensional subspace L. Then f is $\mathcal{H}_n^0$-convex at any point $x^0 \in L$, which follows from Theorem 9. Indeed, when $x^0 \neq 0$, as it was clarified in the proof of Theorem 12 condition (8) coincides with condition $\mathbb{C}(f^{x^0,c,x^0}, x^0, x^0, \zeta)$ with $c = f(x^0) - \varepsilon$ and $\zeta = f(x^0) - f(0)$. When $x^0 = 0$ we can choose $\bar{u} \in L \setminus \{0\}$. Then condition (8) is actually condition $\mathbb{C}(f^{0,c,\bar{u}}, 0, \bar{u}, \zeta)$ with $c = f(0) - \varepsilon$ and $\zeta = f(\bar{u}) - f(0)$.

Let $x^0 \in \mathbb{R}^n$ and $c < f(x^0)$. Suppose that L is a 1-dimensional subspace passing through x^0 and $f|_L$ is not affine. Fix $u \in L \setminus \{0\}$. We claim that

$$\inf_{\lambda>0} \frac{f(x^0 - \lambda u) - c}{\lambda} + \inf_{\mu>0} \frac{f(x^0 + \mu u) - c}{\mu} > 0\,.$$

Observe first, that when λ, $\mu > 0$ it holds

$$\frac{f(x^0 - \lambda u) - c}{\lambda} + \frac{f(x^0 + \mu u) - c}{\mu} > 0\,. \tag{10}$$

Indeed, from the convexity of $f|_L$ we have

$$\frac{\mu}{\lambda+\mu}\, f(x^0 - \lambda u) + \frac{\lambda}{\lambda+\mu}\, f(x^0 + \mu u) \geq f(x^0) > c\,.$$

Multiplying both sides by $\frac{\lambda+\mu}{\lambda\mu}$ we obtain (10).

Now we prove that for any α, $\beta > 0$ it holds

$$\inf_{0<\lambda\leq\alpha} \frac{f(x^0 - \lambda u) - c}{\lambda} + \inf_{0<\mu\leq\beta} \frac{f(x^0 + \mu u) - c}{\mu} > 0\,. \tag{11}$$

Let $f(x^0) > \bar{c} > c$. Since f is lsc, there exists δ with $0 < \delta < \min(\alpha, \beta)$, such that $f(x^0 + tu) \geq \bar{c}$ when $|t| \leq \delta$. Now

$$\inf_{0<\lambda\leq\delta} \frac{f(x^0 - \lambda u) - c}{\lambda} + \inf_{0<\mu\leq\delta} \frac{f(x^0 + \mu u) - c}{\mu} \geq 2\,\frac{\bar{c} - c}{\delta} > 0\,.$$

It remains to show that

$$\inf_{\delta\leq\lambda\leq\alpha} \frac{f(x^0 - \lambda u) - c}{\lambda} + \inf_{\delta\leq\mu\leq\beta} \frac{f(x^0 + \mu u) - c}{\mu} > 0\,. \tag{12}$$

The lower semicontinuity of f gives that the infima

$$\inf_{\delta\le\lambda\le\alpha} \frac{f(x^0-\lambda u)-c}{\lambda} \quad \text{and} \quad \inf_{\delta\le\mu\le\alpha} \frac{f(x^0+\mu u)-c}{\mu}$$

are attained respectively for some $\lambda = \bar\lambda \in [\delta,\, \alpha]$ and $\mu = \bar\mu \in [\delta,\, \beta]$. Hence (12) follows straightforward from (10).

If $f(x^0+\mu u) = +\infty$ for all $\mu \ge \beta$, then for $\lambda > 0$ and $\mu \ge \beta$ the left hand side of (10) becomes $+\infty$. The same is true for $\lambda \ge \alpha$ and $\mu > 0$ provided $f(x^0-\lambda u) = +\infty$ for $\lambda \ge \alpha$.

So, assuming, on the contrary, that (11) is not true, we should have sequences $\lambda_k \to +\infty$, $\mu_k \to +\infty$, $\varepsilon_k \to 0^+$, with $x^0 - \lambda_k u \in \operatorname{dom} f$ and $x^0 + \mu_k u \in \operatorname{dom} f$, such that

$$\varepsilon_k > \frac{f(x^0-\lambda_k u)-c}{\lambda_k} + \frac{f(x^0+\mu_k u)-c}{\mu_k} > 0\,. \tag{13}$$

Since $f|_L$ is convex, we have

$$+\infty > \frac{\mu_k}{\lambda_k+\mu_k}\, f(x^0-\lambda_k u) + \frac{\lambda_k}{\lambda_k+\mu_k}\, f(x^0+\mu_k u) \ge f(x^0)\,,$$

which shows that $x^0 \in \operatorname{dom} f$. Now (13) can be written in the form

$$\varepsilon_k > \frac{f(x^0-\lambda_k u)-f(x^0)}{\lambda_k} + \frac{f(x^0+\mu_k u)-f(x^0)}{\mu_k} + (f(x^0)-c)\left(\frac{1}{\lambda_k}+\frac{1}{\lambda_k}\right) > 0\,.$$

Passing to the limit with $k \to \infty$ we get

$$\lim_k \frac{f(x^0-\lambda_k u)-f(x^0)}{\lambda_k} + \lim_k \frac{f(x^0+\mu_k u)-f(x^0)}{\mu_k} = 0\,. \tag{14}$$

The limits in (14) exist because of the monotonicity properties of the increments for convex functions. On the other hand, since $f|_L$ is convex but not affine, we should have for some k_0

$$\gamma := \frac{f(x^0-\lambda_{k_0} u)-f(x^0)}{\lambda_{k_0}} + \frac{f(x^0+\mu_{k_0} u)-f(x^0)}{\mu_{k_0}} > 0\,.$$

Again from the monotonicity properties of the increments for convex functions, we get that the left hand side in (14) is not less than γ, a contradiction.

The proved result implies the existence of $\zeta \in \mathbb{R}$ such that

$$-\inf_{\lambda>0} \frac{f(x^0-\lambda u)-c}{\lambda} < \zeta < \inf_{\mu>0} \frac{f(x^0+\mu u)-c}{\mu}\,.$$

Now we verify that the hypotheses of Theorem 9 are satisfied. Actually, we prove that condition $\mathbb{C}(g, x^0, u, \zeta)$ holds with $g = f^{x^0,c,u}$ (when $x^0 \ne 0$ we put $u = x^0$).

Assume that condition $\mathbb{C}(g, x^0, u, \zeta)$ does not hold. This means that there is a sequence $x^k \in \mathbb{R}^n$ and a sequence $t_k \in \mathbb{R}$, such that

$$\frac{\tilde g_{x^0,u,\zeta}(x^k) - \tilde g_{x^0,u,\zeta}(x^0+t_k u)}{\|x^k - x^0 - t_k u\|} \to -\infty\,. \tag{15}$$

Without loss of generality we may assume $x^k \notin L$ (if all $x^k \in L$ the above sequence of fractions should have been bounded below). Therefore (15) can be written in the form

$$\frac{f(x^k) - c - \zeta t_k}{\|x^k - x^0 - t_k u\|} \to -\infty\,. \tag{16}$$

We may assume that the numerators are negative for all k. Possibly passing to a subsequence, we may assume that either $\{x^k\}$ is a bounded sequence, or $\|x^k\| \to \infty$. We consider the two possibilities.

1^0. Let $\{x^k\}$ be bounded and suppose that $\|x^k - x^0\| \le r_1$. Since f is lsc, $f(x^k) \ge \mu := \inf\{f(x) \mid \|x - x^0\| \le r_1\} > -\infty$. The sequence $\{t_k\}$ in (16) must be bounded. Indeed, if $|t_k| \to \infty$ we would have in (16)

$$\frac{f(x^k) - c - \zeta t_k}{\|x^k - x^0 - t_k u\|} \ge \frac{\mu - c - |\zeta t_k|}{\|t_k u\| - r_1} \to -\,\frac{|\zeta|}{\|u\|}\,,$$

that is the limit in (16) could not be $-\infty$.

So, let $|t_k| \le r_2$. By the choice of c and ζ we have $f(x^0 + tu) - c - \zeta t > 0$ for $|t| \le r_2$. From the lower semicontinuity of f we get

$$\inf_{|t|\le r_2} (f(x^0 + tu) - c - \zeta t) = \varepsilon > 0\,.$$

Now we prove that there exists $\delta > 0$ such that $\|x - x^0 - tu\| \le \delta$ and $|t| \le r_2$ implies $f(x) - c - \zeta t > 0$ (the existence of $\delta = \delta(t)$ is implied by the lower semicontinuity, but here we must show that δ can be chosen independently on t). If this were not true, there should exist sequences $\{\bar t_k\}$ and $\{\bar x^k\}$ with $|\bar t_k| \le r_2$, $\|\bar x^k - x^0 - \bar t_k u\| \le \frac{1}{k}$, such that $f(\bar x^k) - c - \zeta \bar t_k \le 0$. We may assume that $\bar t_k \to \bar t^0$ with $|\bar t^0| \le r_2$. Now $\bar x^k \to x^0 + \bar t^0 u$. From the lower semicontinuity of f we have

$$f(x^0 + \bar t^0 u) - c - \zeta \bar t^0 \le \liminf(f(\bar x^k) - c - \zeta \bar t_k) \le 0\,,$$

which contradicts to $f(x^0 + \bar t^0 u) - c - \zeta \bar t^0 \ge \epsilon > 0$. The proved property shows that in (16) we should have $\|x^k - x^0 - t_k u\| \ge \delta > 0$, whence

$$\frac{f(x^k) - c - \zeta t_k}{\|x^k - x^0 - t_k u\|} \ge \frac{f(x^k) - c - \zeta t_k}{\delta} \ge \frac{\mu - c - |\zeta| r_2}{\delta}\,.$$

Thus, the limit in (16) cannot be $-\infty$, which shows that this case is impossible.

2^0. Let $\|x^k\| \to +\infty$. Passing to a subsequence, we may suppose that $x^k/\|x^k\| \to v$. We consider separately the cases $v \notin L$ and $v \in L$.

a) The case $v \notin L$. Now there exists $\gamma > 0$ such that the set

$$S_\gamma = \{w \in \mathbb{R}^n \mid \|w\| = 1,\ \|w - v\| \le \gamma\}$$

does not intersect L. Further, applying $0 \in \operatorname{dom} f$ (for the first time), the lower semicontinuity, and the CAL property of f, we show that there exist constants a and b such that

$$f(x^k) \ge a\,\|x^k\| + b \tag{17}$$

for all sufficiently large k. Observe first that $v \in \operatorname{dom} f$. This is true because $f(v) \leq \liminf_k f(x^k/\|x^k\|) < +\infty$. The second inequality is a consequence of

$$f(\frac{x^k}{\|x^k\|}) \leq (1 - \frac{1}{\|x^k\|})\, f(0) + \frac{1}{\|x^k\|}\, f(x^k) \leq |f(0)| < +\infty$$

(observe that (16) implies $f(x^k) < 0$ for all sufficiently large k). From the lower semicontinuity $\alpha := \inf\{f(w) \mid w \in S_\gamma\} > -\infty$. The CAL property implies

$$f(w) \geq \|w\|\,(\alpha - f(0)) + f(0) \quad \text{for} \quad \|w\| \geq 1\,.$$

If

$$\beta = \inf\{f(w) - f(0) - \|w\|(\alpha - f(0)) \mid w = 0, \text{ or } w \neq 0 \text{ and } w/\|w\| \in S_\gamma\}\,,$$

then for $a = \alpha - f(0)$ and $b = \beta + f(0)$ and all $w \neq 0, w/\|w\| \in S_\gamma$, it holds $f(w) \geq a\|w\| + b$, that is (17) is satisfied for all sufficiently large k. Changing eventually a with $-|a|$ we may assume that $a < 0$.

Since $x^0 \in L$, we have $x^0 = \tau u$ for some $\tau \in \mathbb{R}$. Now

$$\begin{aligned} \frac{f(x^k) - c - \zeta t_k}{\|x^k - x^0 - t_k u\|} &= \frac{f(x^k) - c + \zeta\tau - \zeta(t_k + \tau)}{\|x^k - (t_k + \tau)u\|} \\ &\geq \frac{a\|x^k\| + b - c + \zeta\tau}{\|x^k\|\,\|\frac{x^k}{\|x^k\|} - \frac{t_k+\tau}{\|x^k\|}\,u\|} - \frac{|\zeta|}{\|u - \frac{1}{t_k+\tau}\,x^k\|}\,. \end{aligned} \tag{18}$$

Due to $a < 0$ the numerator of the first fraction is negative for sufficiently large k. The last fraction assumes $t_k + \tau \neq 0$. It is easy to verify that the following estimations hold also in the case $t_k + \tau = 0$. Let

$$d_1 = \operatorname{dist}(S_\gamma, L) = \inf\{\|w - x\| \mid w \in S_\gamma,\, x \in L\}\,,$$

$$d_2 = \operatorname{dist}(u, \operatorname{cone} S_\gamma) = \inf\{\|u - w\| \mid w = t\bar{w},\, \bar{w} \in S_\gamma,\, t \geq 0\}\,.$$

Obviously $d_1 > 0$ and $d_2 > 0$. Now (18) gives

$$\frac{f(x^k) - c - \zeta t_k}{\|x^k - x^0 - t_k u\|} \geq \frac{a\|x^k\| + b - c + \zeta\tau}{\|x^k\|\, d_1} - \frac{|\zeta|}{d_2} \to \frac{a}{d_1} - \frac{|\zeta|}{d_2}\,.$$

Therefore the limit in (16) cannot be $-\infty$, a contradiction showing that this case is impossible.

b) The case $v \in L$. Suppose $v = u/\|u\|$ (the case $v = -u/\|u\|$ is investigated similarly). Observe first that $tu \in \operatorname{dom} f$ for all $t \geq 0$. Indeed, we have $t\frac{\|u\|}{\|x^k\|}x^k \to tu$, and since f is lsc and CAL, we have

$$f(tu) \leq \liminf_k f(\frac{t\|u\|}{\|x^k\|}\, x^k) \leq |f(0)| < +\infty$$

because

$$f(\frac{t\|u\|}{\|x^k\|}\, x^k) \leq \left(1 - \frac{t\|u\|}{\|x^k\|}\right) f(0) + \frac{t\|u\|}{\|x^k\|}\, f(x^k) \leq |f(0)|$$

(we have used here $f(x^k) < 0$, true for sufficiently large k due to (16)). Putting $x^0 = \tau u$, we have

$$\lim_{t\to+\infty} \frac{f(tu - f(0))}{t} = \lim_{t\to+\infty} \left(\frac{f(x^0 + (t-\tau)u) - c}{t} \cdot \frac{t-\tau}{t} + \frac{c - f(0)}{t} \right)$$

$$= \lim_{t\to+\infty} \frac{f(x^0 + tu) - c}{t} \geq \inf_{t\to+\infty} \frac{f(x^0 + tu) - c}{t} > \zeta\,.$$

We can fix $t_0 > 0$ and $\varepsilon > 0$ such that $\dfrac{f(t_0 u) - f(0)}{t_0} > \zeta + \varepsilon$. Since f is lsc, there exists a neighborhood W of $t_0 u$ such that

$$\frac{f(w) - f(0)}{t_0} > \zeta + \varepsilon \quad \text{for} \quad w \in W\,.$$

Applying this inequality with $w = \frac{t_0\|u\|}{\|x^k\|}\, x^k$ and the CAL property we get

$$f(x^k) \geq \frac{\|x^k\|}{t_0\|u\|}\, f(\frac{t_0\|u\|}{\|x^k\|}\, x^k) - \left(\frac{\|x^k\|}{t_0\|u\|} - 1 \right) f(0) \geq \frac{\|x^k\|}{\|u\|} (\zeta + \varepsilon) + f(0)\,,$$

whence

$$\begin{aligned} \frac{f(x^k) - c - \zeta t_k}{\|x^k - x^0 - t_k u\|} &\geq \frac{\frac{\|x^k\|}{\|u\|}(\zeta + \varepsilon) + f(0) - c - \zeta t_k}{\|x^k - x^0 - t_k u\|} \\ = \frac{f(0) - c + \zeta\tau}{\|x^k - x^0 - t_k u\|} &+ \frac{1}{\|u\|} \cdot \frac{\zeta\,\left(\|x^k\| - (\tau + t_k)\|u\|\right) + \varepsilon\|x^k\|}{\|x^k - (\tau + t_k)u\|} \\ &\geq \frac{1}{\|u\|} \cdot \frac{\zeta\,\left(\|x^k\| - (\tau + t_k)\|u\|\right) + \varepsilon\|x^k\|}{\|x^k - (\tau + t_k)u\|} \\ &= \frac{1}{\|u\|} \cdot \frac{\zeta\,\left(1 - (\tau + t_k)\frac{\|u\|}{\|x^k\|}\right) + \varepsilon}{\|\frac{x^k}{\|x^k\|} - \frac{\tau + t_k}{\|x^k\|}\, u\|}\,. \end{aligned}$$

We have used $f(0) - c + \zeta\tau \geq 0$, a consequence of the CAL property of f. Since $\dfrac{x^k}{\|x^k\|} = \dfrac{u}{\|u\|} + w^k$ where $w^k \to 0$, we get

$$\frac{f(x^k) - c - \zeta t_k}{\|x^k - x^0 - t_k u\|} \geq \frac{1}{\|u\|} \cdot \frac{\zeta\,\left(\frac{1}{\|u\|} - \frac{\tau + t_k}{\|x^k\|}\right)\|u\| + \varepsilon}{\|\left(\frac{1}{\|u\|} - \frac{\tau + t_k}{\|x^k\|}\right) u + w^k\|}\,. \tag{19}$$

The following cases may occur:

i) The case $\dfrac{1}{\|u\|} - \dfrac{\tau + t_k}{\|x^k\|} \to 0$. Then (19) gives that the left hand side is nonnegative for all sufficiently large k, a contradiction to (16).

ii) The case $s_k := \dfrac{1}{\|u\|} - \dfrac{\tau + t_k}{\|x^k\|} \not\to 0$. Passing to a subsequence we may assume that s_k does not change the sign, and $|s_k| > \delta$ for some $\delta > 0$. Now (19) gives

$$\frac{f(x^k) - c - \zeta t_k}{\|x^k - x^0 - t_k u\|} \geq \frac{\zeta\,\sigma}{\|u\|} \cdot \frac{\|u\|}{\|u + \frac{1}{s_k}\,w^k\|} \to \frac{\zeta\,\sigma}{\|u\|}$$

where $\sigma = \operatorname{sign} s_k$. We get again contradiction with (16).

Thus, any case leads to a contradiction, which shows that condition $\mathbb{C}(g, x^0, u, \zeta)$ has place. Therefore from Theorem 9 we get that f is $\mathcal{H}_n^0$-convex. ■

Acknowledgment. The authors are grateful the anonymous referee for the detailed remarks and suggestions.

References

[1] G. P. Crespi, I. Ginchev, Rocca M., A. Rubinov: Convex along lines functions and abstract convexity. Part I, J. Convex Anal. 14 (2007), no. 1, 185–204.

[2] D. Pallaschke, S. Rolewicz: Foundation of mathematical optimization (Convex analysis without linearity), Kluwer Academic Publishers, Dordrecht, 1997.

[3] R. T. Rockafellar: Convex analysis, Princeton University Press, Princeton, 1970.

[4] R. T. Rockafellar, R. J.-B. Wets: *Variational analysis*, Springer, Berlin, 1998.

[5] A. M. Rubinov: Abstract convexity and global optimization, Kluwer Academic Publishers, Dordrecht, 2000.

[6] A. M. Rubinov, A. P. Shveidel: *Radiant and star-shaped functions*, Pac. J. Optim. 3 (2007), no. 1, 193–212.

[7] A. P. Shveidel: Abstract convex sets with respect to the class of general min-type functions, Optimization 52 (2003), 571–580.

[8] A. P. Shveidel: Further investigation of abstract convexity with respect to the class of general min-type functions, Optimization 56 (2007), 129–147.

[9] I. Singer: *Abstract convex analysis,* Wiley-Interscience Publication, New York, 1997.

LISTA DEI WORKING PAPER 2012

1. Resmini, L., Siedschlag, I., Is Foreign Direct Investment in China Crowding Out the Foreign Direct Investment in other Countries? , Settembre 2012.
2. Crespi, G. P., Ginchev, I., Rocca, M., Robunov, A., Convex along lines functions and abstract convexity. Part II , Ottobre 2012.

UNIVERSITÀ DELLA
VALLE D'AOSTA
UNIVERSITÉ DE LA
VALLÉE D'AOSTE

Dipartimento di
Scienze Economiche e Politiche
Département des
Sciences Économiques et Politiques