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Weak Minimizers, Minimizers and Variational Inequalities for set valued Functions. A blooming wreath?

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Weak Minimizers, Minimizers and Variational Inequalities for set valued Functions. A blooming wreath?

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Abstract

In [10], necessary and sufficient conditions in terms of variational inequalities are introduced to characterize minimizers of convex set valued functions with values in a conlinear space. Similar results are proved in [11, 9] for a weaker concept of minimizers and weaker variational inequalities. The implications are proved using scalarization techniques that eventually provide original problems, not fully equivalent to the set-valued counterparts. Therefore, we try, in the course of this note, to close the network among the various notions proposed. More specifically, we prove that a minimizer is always a weak minimizer, and a solution to the stronger variational inequality always also a solution to the weak variational inequality of the same type. As a special case we obtain a complete characterization of efficiency and weak efficiency in vector optimization by set-valued variational inequalities and their scalarizations. Indeed this might eventually prove the usefulness of the set-optimization approach to renew the study of vector optimization.

1 Introduction

Scalar variational inequalities (for short, VI) apply to study a wide range of problems, such as equilibrium and optimization problems, see e.g. [2], [25]. Generalizations toward vector VI were initiated in [15]; for recent results and survey on this field see [16], [17], [26], [27]. A major peculiarity in vector valued inequalities is the necessity to introduce at least two different solution concepts, e.g. a strong and a weak one. This approach seems to be most natural if referred to vector optimization efficiency and weak efficiency notions.

The notion of differentiable variational inequality arises in the scalar case, when the operator involved in a VI has a primitive function. This kind of VI is widely studied because of its relation to optimization problems. Under mild continuity assumptions, scalar Minty VI (MVI, [28], [33]) of differential type provide a sufficient optimality condition to the primitive optimization problem (a result popularized as Minty variational principle), while scalar Stampacchia VI (SVI, [37]) is only necessary. Assuming some convexity on the primitive function (or monotonicity of the derivative) both VIs are necessary and sufficient optimality conditions. In [5], under generalized differentiability assumptions, scalar Minty VI have been

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studied and it has been proved that the existence of a solution to such a problem implies some regularity property on the primitive optimization problem.

The same approach has been proposed by Giannessi [15] for vector optimization. In his seminal paper, Giannessi studied the relations between a Stampacchia type vector variational inequality and weak efficient solutions of the primitive vector optimization problem. It has been proved that the scalar relations hold old under stronger assumption in the vector case, namely convexity plays a bigger role in the proof. Further researches tried to extend the result to efficient solution, providing a stronger version of the variational inequality and highlighting some peculiarities of the vector case unknown for scalar functions. More recently, also the Minty variational principle has been studied and extended to the vector case. The problem has been posed by Giannessi in [16], where the links between Minty variational inequalities and vector optimization problems were investigated both for efficient and weak efficient solutions. More recently, in [6], [38], some generalization of the vector principle have been proposed in conjuction with weak efficient solutions. In [16], [38], the case of a differentiable objective function f with values in $\mathbb{R}^m$ and a Pareto ordering cone has been studied, proving a vector Minty variational principle for pseudoconvex functions. In [6] a similar result has been extended to the case of an arbitrary ordering cone and a nondifferentiable objective function. Overall, the existing literature pictures a wreath of relations, ranging between weak and strong vector valued inequalities and weak efficiency and efficiency. Some of these relations occur only under (generalized) convexity assumptions, some of the branches of the wreath cannot be fixed.

Although optimization of set-valued functions has been a fast growing topic over the past decades, very few has been proposed about variational inequalities to characterize minimality.

Since the first results by Corley [3], [4] and Dinh The Luc [31], based on a vector optimization approach, several papers have been proposed to provide optimality conditions. Nevertheless, the main approach to derivatives (and therefore to the core of a variational inequality) has been far distant form the basic differential quotient method adopted for scalar (and vector) problems. More recently, a new paradigm, known as set-optimization, has been proposed, compare [20],[23], [29], [30]. In this framework, the very concept of optimal solutions has been thought anew, together with operations among sets, now elements of a complete ordered conlinear space. This leads to overcome some drawbacks in previous attempt to provide variational inequalities for set-valued optimization problems (see e.g. [7]).

In [9] and [11], a notion of weak minimality for set-optimization is presented, motivated by its relation with standard weak efficiency in vector optimization. Under certain regularity assumptions it is proven in [9] that the solutions of the Minty type inequality are weak minimizers of the primitive set-optimization problem. Under slightly weaker assumptions, a weak minimizer of the set-optimization problem solves the Stampacchia differential variational inequality. Under convexity assumptions on the scalarizations, the reverse implications has been proven in [11]. In [8] and [10], a corresponding chain of implications has been provided for minimizers, actually for solutions of set optimization problems, and the corresponding Minty and Stampacchia type differential variational inequalities.

The aim of this paper is to weave lose branches from the previous studies to propose a wreath between set-optimization and set-valued variational inequalities, connecting strong notions in [10] with their weak counterparts presented in [11]. As a special case of our results, we obtain a wreath containing vector optimization efficient and weak efficient solutions.

The paper is organized as follows. We present the general setting of the problem and the basic notation and assumption in Section 2, where some details on conlinear spaces are recalled. In Section 3 we introduce the notion of minimizer and weak minimizer in set-optimization, as well as the scalarization technique that is used to prove main results. The variational inequalities introduced in [11] and [10] are also recalled together with the chains of implications proved in these papers. Section 4 completes the wreath with the main results proving the missing implications.

When of interest, counterexamples are included to show that assumptions cannot be relaxed. The chains of implication provided in each section are illustrated by diagrams.

2 Basics

Throughout the paper, X and Z are real vector spaces, Z locally convex and Hausdorff with topological dual Z^*. The set $\mathcal{U}$ is the set of all closed, convex and balanced 0 neighborhoods in Z, that is a 0–neighborhood base of Z. By $\operatorname{cl} A$, $\operatorname{co} A$ and $\operatorname{int} A$, we denote the closed or convex hull of a set $A \subseteq Z$ and the topological interior of A, respectively. The conical hull of a set A is $\operatorname{cone} A = \{ta \mid a \in A,\, 0 < t\}$.

The set Z is preordered by a closed convex cone $C \neq Z$ with nonempty topological interior, $\operatorname{int} C \neq \emptyset$ by means of $z_1 \leq_C z_2$, if $z_2 \in \{z_1\} + C$. The (negative) dual cone of C is the set $C^- = \{z^* \in Z^* \mid \forall z \in C : \, z^*(z) \leq 0\}$. Since $\operatorname{int} C \neq \emptyset$, there exists a weak* compact base B^* of C^-, i.e. a convex subset with $C^- \setminus \{0\} = \operatorname{cone} B^*$ with $z^*, tz^* \in B^*$ implying $t = 1$ and any net in B^* has a weak* convergent subnet, compare [1, Theorem 1.5.1]

In the sequel we consider the family of subsets of Z

$$\mathcal{G}(Z,C) = \{A \in \mathcal{P}(Z) | A = \operatorname{cl}\operatorname{co}(A + C)\}$$

According to the order relations

$$A \preccurlyeq B \quad \text{iff} \quad B \subseteq A \quad \forall A, B \in \mathcal{G}(Z,C)$$

the set $(\mathcal{G}(Z,C), \supseteq)$ is order complete.Indeed, for any subset $\mathcal{A} \subseteq \mathcal{G}(Z,C)$ it holds

$$\inf \mathcal{A} = \operatorname{cl}\operatorname{co} \bigcup_{A \in \mathcal{A}} A; \quad \sup \mathcal{A} = \bigcap_{A \in \mathcal{A}} A.$$

assuming, by definition that when $\mathcal{A} = \emptyset$ we have $\inf \mathcal{A} = \emptyset$ and $\sup \mathcal{A} = Z$. Particularly, $\mathcal{G}(Z,C)$ possesses a smallest element $\inf \mathcal{G}(Z,C) = Z$ and a greatest one $\sup \mathcal{G}(Z,C) = \emptyset$.

We can also introduce operations on $\mathcal{G}(Z,C)$, defining

$$\begin{aligned} \forall A, B \in \mathcal{G}(Z,C): \quad & A \oplus B = \operatorname{cl}\{a + b \in Z \mid a \in A,\, b \in B\}; \\ \forall A \in \mathcal{G}(Z,C),\, \forall 0 < t: \quad & t \cdot A = \{ta \in Z \mid a \in A\}; \quad 0 \cdot A = C, \end{aligned}$$

The resulting space $\mathcal{G}^\triangle = (\mathcal{G}(Z,C), \oplus, \cdot, C, \preccurlyeq)$ is endowed with neutral element C, $\emptyset$ dominates the addition and $0 \cdot \emptyset = 0 \cdot Z = C$. Moreover,

$$\forall \mathcal{A} \subseteq \mathcal{G}(Z,C),\, \forall B \in \mathcal{G}(Z,C): \quad B \oplus \inf \mathcal{A} = \inf \{B \oplus A \mid A \in \mathcal{A}\},$$

or, equivalently, the inf–residual $A \mathbin{-\!\!^{\centerdot}} B = \inf\{M \in \mathcal{G}(Z,C) \mid A \preccurlyeq B \oplus M\}$ exists for all $A, B \in \mathcal{G}(Z,C)$. It holds (compare [19, Theorem 2.1])

$$\begin{aligned} A \mathbin{-\!\!^{\centerdot}} B &= \{z \in Z \mid B + \{z\} \subseteq A\}; \\ A &\preccurlyeq B \oplus (A \mathbin{-\!\!^{\centerdot}} B). \end{aligned}$$

Overall, the structure of $\mathcal{G}^\triangle$ is that of an order complete inf–residuated conlinear space.

Definition 2.1 *A nonempty set Y together with two algebraic operations $\dotplus : Y \times Y \to Y$ and $\cdot : \mathbb{R}_+ \times Y \to Y$ is called a conlinear space with neutral element θ provided that*
(C1) $(Y, \dotplus, \theta)$ is a commutative monoid with neutral element θ,
(C2) The operations are compatible: (i) $\forall y_1, y_2 \in Y$, $\forall r \in \mathbb{R}_+$: $r \cdot (y_1 \dotplus y_2) = r \cdot y_1 \dotplus r \cdot y_2$, (ii) $\forall y \in Y$, $\forall r, s \in \mathbb{R}_+$: $s \cdot (r \cdot y) = (rs) \cdot y$, (iii) $\forall y \in Y$: $1 \cdot y = y$, (iv) $\forall y \in Y$: $0 \cdot y = \theta$.

A conlinear space $(Y, \dotplus, \cdot, \theta)$ together with a order relation $\preccurlyeq$ on Y is called partially ordered, lattice ordered or order complete conlinear space provided that $(Y, \preccurlyeq)$ has the respective structure and the order is compatible with the algebraic operations $\dotplus$ and $\cdot$:
(C3) (i)$\forall y, y_1, y_2 \in Y$, $y_1 \preccurlyeq y_2$ imply $y_1 \dotplus y \preccurlyeq y_2 \dotplus y$, and (ii) $\forall y_1, y_2 \in Y$, $y_1 \preccurlyeq y_2$, $r \in \mathbb{R}_+$ imply $r \cdot y_1 \preccurlyeq r \cdot y_2$.

A partially ordered conlinear space $(Y, \dotplus, \cdot, \theta, \preccurlyeq)$ is called inf*–residuated, when for all $v, y \in Y$ the element $y \dot{-} v = \inf \{u \in Y \mid y \preccurlyeq v \dotplus u\}$ exists. In this case, $y \dot{-} v$ is called the* inf*–residual of y and v.*

We refer to [12, 13, 14, 19, 20, 32] for a more thorough study of this structure. For the sake of completeness, we recall that it can be proven that a partially ordered conlinear space is inf–residuated, if and only if for all $y \in Y$ and all $A \subseteq Y$ such that $\inf A$ exists, it holds $(y \dotplus \inf \mathcal{A}) = \inf \{y \dotplus a \mid a \in A\}$, compare [19, Theorem 2.1]. The structure of conlinear space may be better understood refereing to the following example.

Example 2.2 *Let us consider $Z = \mathbb{R}$, $C = \mathbb{R}_+$. Then $\mathcal{G}(Z, C) = \{[r, +\infty) \mid r \in \mathbb{R}\} \cup \{\mathbb{R}\} \cup \{\emptyset\}$, and $\mathcal{G}^{\triangle}$ can be identified (with respect to the algebraic and order structures which turn $\mathcal{G}(\mathbb{R}, \mathbb{R}_+)$ into an ordered conlinear space and a complete lattice admitting an inf-residuation) with $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ using the 'inf-addition' $\dotplus$ (see [19, 34]). The inf-residuation on $\overline{\mathbb{R}}$ is given by*

$$r \dot{-} s = \inf \left\{t \in \mathbb{R} \mid r \leq s \dotplus t\right\}$$

for all $r, s \in \overline{\mathbb{R}}$, compare [19] for further details.

Basic notions from real analysis can be easily extended to set–valued functions mapping onto the conlinear space $\mathcal{G}^{\triangle}$. For instance a function $f \colon X \to \mathcal{G}^{\triangle}$ is called convex when

$$\forall x_1, x_2 \in X,\ \forall t \in (0, 1) : f(tx_1 + (1 - t)x_2) \preccurlyeq tf(x_1) \dotplus (1 - t) f(x_2).$$

Moreover f is called positively homogeneous when

$$\forall 0 < t, \forall x \in X \colon f(tx) \preccurlyeq tf(x),$$

and it is called sublinear if it is positively homogeneous and convex. As a standard notation, we refer to the image set of a subset $A \subseteq X$ through f by $f[A] = \{f(x) \in W \mid x \in A\} \subseteq \mathcal{G}^{\triangle}$ and to the (effective) domain of a function $f : X \to \mathcal{G}^{\triangle}$ is the set $\operatorname{dom} f = \{x \in X \mid f(x) \neq \sup \mathcal{G}^{\triangle}\}$. A function $f : X \to \mathcal{G}^{\triangle}$ is called proper, if $\operatorname{dom} f \neq \emptyset$ and $\inf \mathcal{G}^{\triangle} \notin f[X]$.

Dealing with set–valued functions $f \colon X \to \mathcal{G}^{\triangle}$, scalarization is a common tool for optimization problems. We first recall that the recession cone of a nonempty closed convex set $A \subseteq Z$ is the closed convex cone $0^+A = \{z \in Z \mid A + \{z\} \subseteq A\}$, compare [39, p.6]. By definition, $0^+\emptyset = \emptyset$ is assumed. If $A \in \mathcal{G}^{\triangle} \setminus \{\emptyset\}$, then $0^+A = A \dot{-} A$ and $C \subseteq 0^+A$ are satisfied. Especially, $\operatorname{int}(0^+A) \neq \emptyset$ and $(0^+A)^- \subseteq C^-$, hence $B^* \cap (0^+A)^-$ is a weak* compact base of $(0^+A)^-$.

Each element of $\mathcal{G}^{\triangle}$ is closed and convex and $A = A + C$, hence by a separation argument we can prove

$$\forall A \in \mathcal{G}^{\triangle}: \quad A = \bigcap_{z^* \in B^*} \left\{ z \in Z \mid -\sigma(z^*|A) \leq -z^*(z) \right\}, \tag{2.1}$$

where $\sigma(z^*|A) = \sup \{z^*(z) \mid z \in A\}$ is the support function of A at z^*. Therefore, $A = \emptyset$ if and only if there exists a $z^* \in B^*$ such that $-\sigma(z^*|A) = +\infty$, or equivalently if the same holds true for all $z^* \in B^*$.

According to this notation, introducing the family of scalarizations for $f \colon X \to \mathcal{G}^{\triangle}$ as the extended real-valued functions $\varphi_{f,z^*} \colon X \to \mathrm{I\!R} \cup \{\pm\infty\}$ defined by

$$\forall z^* \in C^- \setminus \{0\}: \quad \varphi_{f,z^*}(x) = \inf \{-z^*(z) \mid z \in f(x)\}$$

we obtain from (2.1) the following representation of f

$$\forall x \in X: \quad f(x) = \bigcap_{z^* \in B^*} \left\{ z \in Z \mid \varphi_{f,z^*}(x) \leq -z^*(z) \right\}.$$

Some properties of f are inherited by its scalarizations and vice versa. For instance, f is convex if and only if φ_{f,z^*} is convex for each $z^* \in B^*$.

To some extent, continuity or its relaxations are a common assumption in variational inequality applications to optimization. The following definition summarize those continuity concepts that are used in the sequel.

Definition 2.3 *(a) Let $\varphi : X \to \overline{\mathrm{I\!R}}$ be a function, $x_0 \in X$. Then φ is said to be lower semicontinuous (l.s.c.) at x_0, iff*

$$\forall r \in \mathrm{I\!R}: \quad r < \varphi_{f,z^*}(x_0) \;\Rightarrow\; \exists U \in \mathcal{U}: \forall u \in U: r < \varphi_{f,z^*}(x_0 + u).$$

(b) A set $\Psi = \left\{ \varphi_i : X \to \overline{\mathrm{I\!R}} \mid i \in I \right\}$ is lower equicontinuous in $x_0 \in \bigcap_{i \in I} \operatorname{dom} \varphi_i$, if

$$\forall \varepsilon > 0 \, \exists W \in \mathcal{U}_X(0) \, \forall x \in x_0 + W \, \forall i \in I: \quad \varphi_i(x_0) \leq \varphi_i(x) + \varepsilon$$

(c) Let $\psi : S \subseteq X \to Z$ be a function, then ψ is called C–continuous at $x_0 \in S$, iff

$$\forall V \in \mathcal{U}(\psi(x_0)) \, \exists U \in \mathcal{U}(x_0) \, \forall x \in S \cap U: \quad \psi(x) \in V + C;$$

(d) Let $F : X \to \mathcal{P}(Z)$ be a function, then F is called upper Hausdorff continuous at $x_0 \in S$, iff

$$\forall V \in \mathcal{U}(0) \, \exists U \in \mathcal{U}(x_0) \, \forall x \in U: \quad F(x) \subseteq F(x_0) + U;$$

(e) Let $f : X \to \mathcal{G}^{\triangle}$ be a function, $M^* \subseteq C^- \setminus \{0\}$. Then f is said M^*– lower semicontiuous (M^*–l.s.c.) at x_0, iff φ_{f,z^*} is l.s.c. at x_0 for all $z^* \in M^*$.

(f) Let $f : X \to \mathcal{G}^{\triangle}$ be a function. If

$$f(x) \preccurlyeq \liminf_{u \to 0} f(x + u) = \bigcap_{U \in \mathcal{U}} \operatorname{cl} \operatorname{co} \bigcup_{u \in U} f(x + u)$$

is satisfied, then f is lattice lower semicontinuous (lattice l.s.c.) at x. A function $f : X \to \mathcal{G}^{\triangle}$ is lattice l.s.c. if and only if it is lattice l.s.c. everywhere.

In [24], it has been proven that if f is $C^- \setminus \{0\}$–l.s.c. at x, then it is also lattice l.s.c. at x. Since we assume $\operatorname{int} C \neq \emptyset$, f is $C^- \setminus \{0\}$–l.s.c. at x if and only if f is $B*$–l.s.c. at x. One can show that if f is convex, then f is lattice l.s.c. if and only if $\operatorname{graph} f = \{(x,z) \mid z \in f(x)\} \subseteq X \times Z$ is a closed set with respect to the product topology, see [21].

In [9, Proposition 2.3] it has been proven that $f : X \to \mathcal{G}^\triangle$ is upper Hausdorff continuous at $x_0 \in \operatorname{dom} F$, if and only if $\Psi = \left\{\varphi_{f,z^*} : X \to \overline{\mathrm{IR}} \mid z^* \in B^*\right\}$ is lower equicontinuous at x_0 in which case f is B^*- lower semicontinuous at x_0 which in turn implies lower lattice continuity at x_0, compare [24] for a detailed study of continuity concepts for set valued functions.

Remark 2.4 *In this paper we mainly refer to $\mathcal{G}^\triangle$–valued functions. However this is not a restriction as any set–valued function $F : X \to \mathcal{P}(Z)$ can be associated to its epigraphical, or $\mathcal{G}^\triangle$, extension given by $F^C : X \to \mathcal{G}^\triangle$ defined by*

$$F^C(x) = \begin{cases} \operatorname{cl}\operatorname{co}\left(F(x) + C\right), & \text{if} \quad F(x) \neq \emptyset \\ F(x) = \emptyset & \qquad \text{elsewhere.} \end{cases}$$

Recalling that C–convexity of F is defined by

$$\forall x, y \in X,\ \forall t \in (0,1): \quad tF(x) + (1-t)F(y) \subseteq F(tx + (1-t)y) + C.$$

we have that F^C is convex if F is C–convex.
Moreover, any vector–valued function $\psi : S \subseteq X \to Z$ can be regarded as a set–valued function on X defined as $F(x) = \{\psi(x)\}$ whenever $x \in S$ and $F(x) = \emptyset$. Therefore we can always associate to a vector–valued function its $\mathcal{G}^\triangle$ extension $\psi^C = F^C : X \to \mathcal{G}^\triangle$.
Obviously, $\operatorname{dom} \psi^C = S$ *and for all $z^* \in B^*$ it holds*

$$\varphi_{\psi^C, z^*}(x) = \begin{cases} -z^*\psi(x) \in \mathrm{IR} & \text{if} \ \ x \in S \\ +\infty & \text{elsewhere.} \end{cases}$$

Therefore ψ is C–continuous at $x \in S$ if and only if $\left\{-z^*\psi : X \to \overline{\mathrm{IR}} \mid z^* \in B^*\right\}$ is lower equicontinuous at x (see e.g. [11, Lemma 2.14]). If additionally the ordering cone C (and hence $C^- \setminus \{0\}$) is polyhedral, then ψ is C–continuous, if and only if ψ^C is B^*-l.s.c. at x (see [31, Corollary 5.6]). Under the same assumption on C, lower semicontinuity of a finite set of scalarizations, namely those with respect to the extreme directions of $C^- \setminus \{0\}$, is known to be equivalent to C–continuity of ψ.

Finally we define the restriction of a set valued function $f : X \to \mathcal{G}^\triangle$ to a segment with end points $x_0, x \in X$ as $f_{x_0,x} : \mathrm{IR} \to \mathcal{G}^\triangle$, given by

$$f_{x_0,x}(t) = \begin{cases} f(x_0 + t(x - x_0)), \text{ if } t \in [0,1]\,; \\ \emptyset, \text{ elsewhere.} \end{cases}$$

Setting $x_t = x_0 + t(x - x_0)$ for all $t \in \mathrm{IR}$, the scalarization of the restricted function $f_{x_0,x}$ is equal to the restriction of the scalarization of f for all $z^* \in C^- \setminus \{0\}$.

An immediate generalization of the results in the remainder of this note is to replace convexity of f by radial convexity of f at x_0, meaning that $f_{x_0,x} : \mathrm{IR} \to \mathcal{G}^\triangle$ is convex for all $x \in X$ and likewise replacing lower semicontinuity by the corresponding radial definition. In [9] and [8], the convexity assumption is dropped and replaced by more general monotonicity assumptions on the scalarization of the set valued function.

3 Minimality and variational inequality formulation

In set–optimization several notions of minimimality can be defined through the order introduced in $\mathcal{G}^{\triangle}$. In this paper we focus on the following definitions that introduce two different notions, a stronger and a weaker one, respectively.

Definition 3.1 *[23] Let $f : X \to \mathcal{G}^{\triangle}$ be a function. Then $x_0 \in \operatorname{dom} f$ is called a* minimizer *of f, if the following holds true.*

$$\forall x \in X : \quad (f(x) \preccurlyeq f(x_0) \; \Rightarrow f(x) = f(x_0))\,. \tag{Min}$$

Definition 3.2 *[11] Let $f : X \to \mathcal{G}^{\triangle}$ be a function. Then $x_0 \in \operatorname{dom} f$ is called a* weak l, scalarized weak *or* weak minimizer *of f, if either $f(x) = Z$, or*

$$\forall x \in X : \quad f(x_0) \nsubseteq \operatorname{int} f(x); \tag{w-l-Min}$$

$$\forall x \in X \, \exists z^* \in B^* : \quad \varphi_{f,z^*}(x_0) \leq \varphi_{f,z^*}(x) \neq -\infty; \tag{w-sc-Min}$$

$$\forall x \in X \, \forall U \in \mathcal{U} : \quad f(x_0) \oplus U \nsubseteq f(x). \tag{w-Min}$$

The chain of implications in Definition 3.2 is (w-l-Min)⇒(w-sc-Min)⇒(w-Min) (see [11, Proposition 2.11]) hence each weak-l-minimizer of f in the sense of [22] is a weak minimizer. Moreover, if $f = F^C : X \to \mathcal{G}^{\triangle}$ and $F(x_0)$ is a compact set, then the three types of weak minimizers coincide [9, Proposition 2.1]. The motivation of our naming lays in the special case $f = \psi^C$. Then $x_0 \in \operatorname{dom} f$ is a weak minimizer of f if and only if $\psi(x_0)$ is a weakly efficient element of $\psi[X]$, i.e. for all $x \in \operatorname{dom} f$ it holds $\psi(x_0) \notin \psi(x) + \operatorname{int} C$. Likewise, x_0 is a minimizer of f if and only if $\psi(x_0)$ is an efficient element of $\psi[X]$, i.e. for all $x \in \operatorname{dom} f$ $\psi(x_0) \in \psi(x) + C$ implies $\psi(x) \in \psi(x_0) + C$.

To introduce a variational inequality associated to the set–optimization of $f : X \to \mathcal{G}^{\triangle}$, we first need a notion of derivative of f. Recent results on scalar and vector Minty type variational inequalities such as [5, 6] have used the concept of (lower) Dini derivative to state the problem. The structure of inf–residuated image space allows to propose such a derivative also for set–valued maps. We had rather present the definition on a general setting, than restricting it to the $\mathcal{G}^{\triangle}$ case that will be applied for the main result. Doing so, we can stress how the next definition allows to extend the Dini derivative of scalar valued functions to extended real valued functions (see e.g. [21, 35]).

Definition 3.3 *Let Y be a* inf*–residuated order complete conlinear space, $f : X \to Y$ and $x, u \in X$. The* upper *and* lower Dini directional derivative *of f at x in direction u are given by*

$$f^{\uparrow}(x,u) = \limsup_{t \downarrow 0} \frac{1}{t}\left(f(x+tu) \mathbin{-^{\!\cdot}} f(x)\right) = \inf_{0<s} \sup_{0<t\leq s} \frac{1}{t}\left(f(x+tu) \mathbin{-^{\!\cdot}} f(x)\right);$$
$$f^{\downarrow}(x,u) = \liminf_{t \downarrow 0} \frac{1}{t}\left(f(x+tu) \mathbin{-^{\!\cdot}} f(x)\right) = \sup_{0<s} \inf_{0<t\leq s} \frac{1}{t}\left(f(x+tu) \mathbin{-^{\!\cdot}} f(x)\right).$$

If both derivatives coincide, then $f'(x,u) = f^{\uparrow}(x,u) = f^{\downarrow}(x,u)$ is the Dini directional derivative *of f at x in direction u.*

As the Dini derivatives are defined 'radially', it is easy to see that following statements hold under radial assumptions, too. For notational simplicity, we refrain from this generalization, hoping to improve the clarity of the general scheme presented.

Remark 3.4 *Definition 3.3 actually provides a generalization of the classical notion of Dini derivative for scalar valued functions. Indeed let $\varphi : X \to \overline{\mathrm{I\!R}}$ be an extended valued scalar function. If $\varphi(x+tu) \in \mathrm{I\!R}$ is satisfied for all $t \in [0, t_0]$ for a given $0 < t_0$, then the differential quotient is real, too, hence in this case the above defined derivatives coincide with the standard definition in the literature, compare [18]. If $x \notin \operatorname{dom}\varphi$, then $\varphi(x+tu) \dot{-} \varphi(x) = -\infty$ for all $t > 0$, so $\varphi'(x,u) = -\infty$. On the other hand, if $\varphi(x) = -\infty$, then $\varphi(x+tu) \dot{-} \varphi(x) = -\infty$, whenever $\varphi(x+tu) = -\infty$ and $\varphi(x+tu) \dot{-} \varphi(x) = +\infty$, else. The value of the derivatives in this case depends on the behavior of φ in a proximity of x.*

The following characterization of the Dini derivative extends a classical result to set-valued functions.

Proposition 3.5 *[11, Proposition 3.4] Let Y be a $\inf$–residuated order complete conlinear space, $f : X \to Y$. If f is convex, then the Dini derivative exists for all $x, u \in X$ and it holds*

$$f'(x,u) = \inf_{0<t} \frac{1}{t} \left(f(x+tu) \dot{-} f(x) \right).$$

Moreover, $f' : X \times X \to Y$ is sublinear in its second component.

If $Y = \mathcal{G}^{\triangle}$, then for all $x, u \in X$ and $0 < s$ the directional derivative of a convex function $f : X \to \mathcal{G}^{\triangle}$ is

$$f'(x,u) = \operatorname{cl} \bigcup_{0<t\leq s} \frac{1}{t} \left(f(x+tu) \dot{-} f(x) \right),$$

the differential quotient is decreasing as t converges towards 0. Moreover, as $\operatorname{int} C \neq \emptyset$ is assumed,

$$\operatorname{int} f'(x,u) = \bigcup_{0<t\leq s} \operatorname{int} \frac{1}{t} \left(f(x+tu) \dot{-} f(x) \right)$$

is satisfied for all $x, u \in X$ and all $0 < s$, compare [11, Lemma 3.5].

Proposition 3.6 *Let $f : X \to \mathcal{G}^{\triangle}$ be a convex, set valued function, then*

$$\liminf_{t\downarrow 0} \frac{1}{t} \left(f(x+tu) \dot{-} f(x) \right) = \bigcap_{0<s} \operatorname{cl} \bigcup_{0<t<s} \frac{1}{t} \left(f(x+tu) \dot{-} f(x) \right),$$

the upper Painleve Kuratowski limit of the differential quotient, compare [31][p. 21].

Proof. Indeed, we only need to check the convexity of the set $\bigcup\limits_{0<t<s} \frac{1}{t} \left(f(x+tu) \dot{-} f(x) \right)$. Let $z_1, z_2 \in \bigcup\limits_{0<t<s} \frac{1}{t} \left(f(x+tu) \dot{-} f(x) \right)$ be given, then there exists $0 < t_1, t_2 < s$ such that

$$f(x) + t_i z_i \subseteq f(x + t_i u)$$

is true for $i = 1, 2$. Let $r \in [0,1]$ be given, $t_0 = (1-r)t_1 + rt_2$. By convexity of the set $f(x)$ it holds

$$f(x) + t_0 \left(\frac{(1-r)t_1}{t_0} z_1 + \frac{rt_2}{t_0} z_2 \right) = (1-r)\left(f(x) + t_1 z_1\right) + r\left(f(x) + t_2 z_2\right),$$

hence by convexity of the function f

$$f(x) + t_0 \left(\frac{(1-r)t_1}{t_0} z_1 + \frac{rt_2}{t_0} z_2 \right) \subseteq (1-r)f(x + t_1 u) + rf(x + t_2 u) \subseteq f(x + t_0 u),$$

implying $\operatorname{co}\{z_1, z_2\} \subseteq \bigcup\limits_{0<t<s} \frac{1}{t}\left(f(x+tu) \dot{-} f(x)\right)$.

$\square$

We need also to use relations between the Dini derivative of the set–valued function and those of its scalarization.

Proposition 3.7 *[10, Proposition 2.36] Let $f : X \to \mathcal{G}^\triangle$ be a convex function, $x, u \in X$. Then*

$$\bigcap_{z^* \in B} \left\{ z \in Z \mid \varphi'_{f,z^*}(x,u) \le -z^*(z) \right\} \preccurlyeq f'(x,u);$$
$$\forall z^* \in C^- \setminus \{0\} : \quad \varphi'_{f,z^*}(x,u) \le -\sigma(z^* | f'(x,u)).$$

Although in general taking the scalarization of the derivative is not equal to the derivative of the scalarization, the equivalence occurs in the special case of the epigraphical extension of a vector valued function.

Proposition 3.8 *[11, Proposition 3.10] Let $\psi : S \subseteq X \to Z$ be a C–convex vector valued function, $f = \psi^C : X \to \mathcal{G}^\triangle$ its epigraphical extension, $x, u \in X$. Then for all $z^* \in C^- \setminus \{0\}$ it holds*

$$\forall z^* \in C^- \setminus \{0\} : \quad -\sigma(z^* | f'(x,u)) = \varphi'_{f,z^*}(x,u). \tag{SR}$$

For a general function $f : X \to \mathcal{G}^\triangle$ if (SR) is satisfied, then also the weaker condition

$$f'(x,u) = \bigcap_{z^* \in B^*} \left\{ z \in Z \mid \varphi'_{f,z^*}(x,u) \le -z^*(z) \right\} \tag{WR}$$

holds true. Again, when $f : X \to \mathcal{G}^\triangle$ is the epigraphical extension of a C–convex vector function $\psi : S \subseteq X \to Z$, then Property (WR) is satisfied.

In the sequel, property (SR) will be refereed to as strong regularity, while property (WR) as weak regularity.

Attempts to characterize minimizers and weak minimizers in set-optimization through varaitional inequalities have been proposed in [8], [10] and [9], [11]. We recall the definitions of Stampacchia and Minty variational inequalities and their scalarizations used in the previous papers.

Definition 3.9 *Let $f : X \to \mathcal{G}^\triangle$ be a convex function, $x_0 \in \operatorname{dom} f$. Then x_0 solves the set valued Stampacchia inequality, if and only if*

$$f(x_0) = Z \ \vee \ \forall x \in \operatorname{dom} f : \ f(x) \neq f(x_0) \ \Rightarrow \ 0 \notin f'(x_0, x - x_0). \tag{SVI_M}$$

Definition 3.10 *Let $f : X \to \mathcal{G}^{\triangle}$ be a convex function, $x_0 \in \operatorname{dom} f$. Then x_0 solves the scalarized Stampacchia inequality, if and only if*

$$f(x_0) = Z \ \vee \ \forall x \in \operatorname{dom} f : \ f(x) \neq f(x_0) \ \Rightarrow \ \exists z^* \in B^* : \ 0 < \varphi'_{f,z^*}(x_0, x - x_0). \qquad (svi_M)$$

Definition 3.11 *Let $f : X \to \mathcal{G}^{\triangle}$ be a convex function, $x_0 \in \operatorname{dom} f$. Then x_0 solves the set valued Minty inequality, if and only if*

$$\forall x \in X : \ f(x) \neq f(x_0) \quad \Rightarrow \quad f'(x, x_0 - x) \not\subseteq 0^+ f(x). \qquad (MVI_M)$$

Definition 3.12 *Let $f : X \to \mathcal{G}^{\triangle}$ be a convex function, $x_0 \in \operatorname{dom} f$. Then x_0 solves the scalarized Minty inequality, if and only if*

$$\forall x \in X : \ f(x) \neq f(x_0) \quad \Rightarrow \quad \exists z^* \in B^* : \quad \varphi_{f,z^*}(x) \neq -\infty \wedge \varphi'_{f,z^*}(x, x_0 - x) < 0. \ (mvi_M)$$

In [10], the following scheme has been proved for convex set valued functions $f : X \to \mathcal{G}^{\triangle}$

Proposition 3.13 *Let $f : X \to \mathcal{G}^{\triangle}$ be a convex function, $x_0 \in \operatorname{dom} f$.*

(a) The following implications hold without further assumptions.

$$(svi_M) \ \Rightarrow \ (SVI_M) \ \Rightarrow \ (Min) \ \Rightarrow \ (mvi_M) \ \Leftarrow \ (MVI_M);$$

(b) If the weak regularity assumption (WR) is satisfied, then $((svi_M) \ \Leftrightarrow \ (SVI_M))$ is true, strong regularity (SR) implies $((mvi_M) \ \Leftrightarrow \ (MVI_M))$

(c) If f is B^*–l.s.c. in x_0 and the set B^* in (mvi_M) can be replaced by a finite subset $M^* \subseteq B^*$, then $((Min) \Leftrightarrow (mvi_M))$ is true.

(d) Especially, if $\psi : S \subseteq X \to Z$ is given, $f(x) = \psi^C(x)$ for all $x \in X$, then

$$(svi_M) \ \Leftrightarrow \ (SVI_M) \ \Rightarrow \ (Min) \ \Rightarrow \ (mvi_M) \ \Leftrightarrow \ (MVI_M)$$

is satisfied. If additionally the ordering cone C is polyhedral and f is B^*–l.s.c. at x_0, then the following scheme is true.

$$(svi_M) \ \Leftrightarrow \ (SVI_M) \ \Rightarrow \ (Min) \ \Leftrightarrow \ (mvi_M) \ \Leftrightarrow \ (MVI_M).$$

The assumption of C polyhedral to prove that (mvi_M) implies (Min) cannot be dropped, as the following example shows.

Example 3.14 *Let* $X = \mathbb{R}$ *and* $Z = l^\infty$ *be given with the usual ordering cone* $C = \{z \in Z \mid \forall n \in \mathbb{N}\, z_n \geq 0\}$*. The function* $f = \psi^C : X \to \mathcal{G}(Z, C)$ *is defined with* $\operatorname{dom}\psi = [-1, 1]$ *and*

$$\forall x \in \operatorname{dom}\psi\, \forall n \in N: \quad (\psi(x))_n = \max\left\{(\sqrt{n^2-1} - n)(x+1), \frac{1}{n - \sqrt{n^2-1}}(x-1)\right\}$$

Then $-e_n^* = (0, ..., 0, -1, 0, ...) \in C^- \setminus \{0\}$ *and* $\varphi_{f,-e_n^*}(x) = (\psi(x))_n$ *is true for all* $n \in \mathbb{N}$ *and all* $x \in [-1, 1]$*. Especially,* f *is convex and radially upper Hausdorff continuous in* 1*. However,*

$$\forall -1 < x < 1: \quad f(1) \subsetneq f(x),$$

while

$$\varphi'_{f,-e_n^*}(x, 1) = \begin{cases} \sqrt{n^2-1} - n, & \text{if } x < \frac{\sqrt{n^2-1}}{n}; \\ \frac{1}{n-\sqrt{n^2-1}}, & \text{if } x \geq \frac{\sqrt{n^2-1}}{n}. \end{cases}$$

As directional derivatives are positively homogeneous, this implies (mvi_M) *is satisfied at* 1*, but* 1 *is not a minimizer of* f*.*

Weaker inequalities can be introduced as well to characterize weak efficiency.

Definition 3.15 *Let* $f : X \to \mathcal{G}^\triangle$ *be a convex function, then* x_0 *solves the weak set valued Stampacchia inequality, if and only if*

$$f(x_0) = Z \ \vee\ \forall x \in X : 0 \notin \operatorname{int} f'(x_0, x - x_0). \tag{SVI_W}$$

Definition 3.16 *Let* $f : X \to \mathcal{G}^\triangle$ *be a convex function,* $x_0 \in \operatorname{dom} f$*. Then* x_0 *solves the weak scalarized Stampacchia inequality, if and only if*

$$f(x_0) = Z \ \vee\ \forall x \in X : \exists z^* \in B^* : 0 \leq \varphi'_{f,z^*}(x_0, x - x_0). \tag{svi_W}$$

Definition 3.17 *Let* $f : X \to \mathcal{G}^\triangle$ *be a convex function,* $x_0 \in \operatorname{dom} f$*. Then* x_0 *solves the weak set valued Minty inequality, if and only if*

$$f(x_0) = Z \ \vee\ \forall x \in X : f'(x, x_0 - x) \nsubseteq \operatorname{int} 0^+ f(x). \tag{MVI_W}$$

Definition 3.18 *Let* $f : X \to \mathcal{G}^\triangle$ *be a convex function,* $x_0 \in \operatorname{dom} f$*. Then* x_0 *solves the weak scalarized Minty inequality, if and only if*

$$f(x_0) = Z \ \vee\ \forall x \in X : \exists z^* \in B^* : \quad \varphi_{f,z^*}(x) \neq -\infty \wedge \varphi'_{f,z^*}(x, x_0 - x) \leq 0. \tag{mvi_W}$$

In [9, 11], the following scheme has been proved for convex set valued functions $f : X \to \mathcal{G}^\triangle$.

Proposition 3.19 *Let* $f : X \to \mathcal{G}^\triangle$ *be a convex function,* $x_0 \in \operatorname{dom} f$*.*

(a) The following implications hold without further assumptions.

$$(svi_W) \Rightarrow (w\text{-}sc\text{-}Min) \Rightarrow (mvi_W) \Leftarrow (MVI_M);$$

$$(svi_W) \Rightarrow (SVI_W) \Leftrightarrow (w\text{-}Min);$$

(b) *If the strong regularity assumption* (SR) *is satisfied, then the following implications hold with equivalence.*

$$(SVI_W) \;\Leftrightarrow\; (svi_W) \;\Leftrightarrow\; (w\text{-}sc\text{-}Min) \;\Leftrightarrow\; (w\text{-}Min)$$

(c) *If the set B^* in* (w-sc-Min) *can be replaced by a finite subset $M^* \subseteq B^*$, then the follwoing equivalence is satisfied.*

$$(svi_W) \Leftrightarrow (w\text{-}sc\text{-}Min)$$

(d) *If the set B^* in* (mvi_W) *can be replaced by a finite subset $M^* \subseteq B^*$ and f is M^*–l.s.c. in x_0, then*

$$(svi_W) \;\Leftrightarrow\; (w\text{-}sc\text{-}Min) \;\Leftrightarrow\; (mvi_W)$$

(e) *If $f = F^C$ with $F(x_0) \subseteq Z$ compact, then*

$$(SVI_W) \;\Leftrightarrow\; (svi_W) \;\Leftrightarrow\; (w\text{-}sc\text{-}Min)\,(\;\Leftrightarrow\; (w\text{-}Min) \;\Leftrightarrow\; (w\text{-}l\text{-}Min))\,.$$

(f) *If $f = F^C$ with $F(x_0) \subseteq Z$ compact, the scalarizations $\varphi^{\triangle}_{f,z^*}$ are proper for all $z^* \in B^*$ and $f_{x_0,x}$ is upper Hausdorff continuous for all $x \in \operatorname{dom} f$, then*

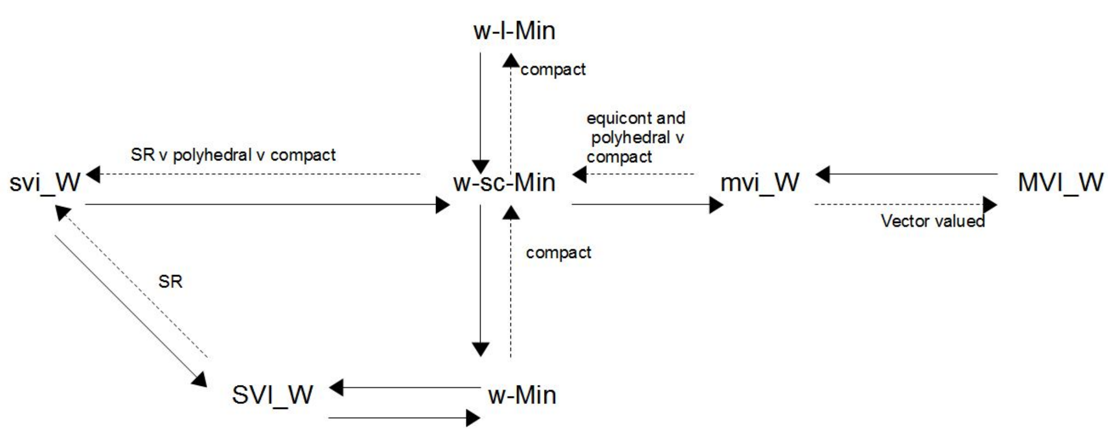
$$(SVI_W) \;\Leftrightarrow\; (svi_W) \;\Leftrightarrow\; (w\text{-}Min) \;\Leftrightarrow\; (mvi_W).$$

(g) *Especially, if $\psi : S \subseteq X \to Z$ is given, $f(x) = \psi^C(x)$ for all $x \in X$, then*

$$(SVI_W) \;\Leftrightarrow\; (svi_W) \;\Leftrightarrow\; (w\text{-}Min) \;\Leftrightarrow\; (w\text{-}sc\text{-}Min) \;\Rightarrow\; (mvi_M) \;\Leftrightarrow\; (MVI_M)$$

is satisfied. If additionally $\psi_{x_0,x}$ is C-continuous for all $x \in S$ or if the ordering cone C is polyhedral and f is radially B^*–l.s.c. at x_0, then all implications are satisfied with equivalence.

$$(SVI_M) \;\Leftrightarrow\; (svi_M) \;\Leftrightarrow\; (w\text{-}Min) \;\Leftrightarrow\; (w\text{-}sc\text{-}Min) \;\Leftrightarrow\; (w\text{-}l\text{-}Min) \;\Leftrightarrow\; (mvi_M) \;\Leftrightarrow\; (MVI_M).$$



In general a solution to the scalarized Minty variational inequality (mvi_M) is not a solution to the set-valued one, as the following example shows.

Example 3.20 *Let $X = \mathrm{I\!R}$, $Z = \mathrm{I\!R}^2$ and $C = \mathrm{I\!R}^2_+$ be given, $x_0 = \frac{2}{3}$ and $f : \mathrm{I\!R} \to \mathcal{G}^\triangle$ with*

$$f(x) = \begin{cases} \left\{ z = (z_1, z_2)^T \in \mathrm{I\!R}^2 \mid z_1 + z_2 \geq (1 - \frac{1}{2}x),\, z_1 \geq x,\, z_2 \geq x \right\}, & \text{if } 0 \leq x \leq \frac{2}{3}; \\ \emptyset, & \text{elsewhere.} \end{cases}$$

Then $f(0) = \left\{(x, 1-x)^T \mid 0 \leq x \leq 1\right\} + C$ is the sum of a compact set and C, f is convex and upper Hausdorff continuous in the domain and each scalarization is proper. Let $z^* = (-1,-1)^T$, then $\varphi^\triangle_{f,z^*}(0) = 1$ and $\left(\varphi^\triangle_{f,z^*}\right)'(0,1) = -\frac{1}{2}$ while $f'(0,1) = (1,1)^T + C \subsetneq \operatorname{int} 0^+ f(0)$. So ($mvi_M$) (and ($mvi_W$)) is satisfied even with a finite subset of B^* but (MVI_M) (and MVI_W) is not satisfied. Especially, (SR) is not satisfied.

4 Main Results

While minimizer clearly are weak minimizer, we still need to prove that the same implication holds between the strong and the weak formulation of the variational inequalities. This result closes the loop between the previous schemes, providing a complete wreath between variational inequalities and minimality.

The following results prove the relations holding between the four couples of inequalities introduced in the previous section.

Proposition 4.1 *Let $f : X \to \mathcal{G}^\triangle$ be a convex function, $x_0 \in \operatorname{dom} f$. If x_0 solves (SVI_M), then it also solves (SVI_W) and $f(x) = f(x_0)$ implies that $f_{x_0,x}$ is constant on $[0,1]$.*

Proof. Assume $f(x) = f(x_0)$ and $f(x_t) \neq f(x)$ for some $t \in (0,1)$. By convexity, $f(x_0) \subsetneq f(x_t)$ is satisfied, hence

$$0 \in f(x) \dot{-} f(x_0) \subseteq f'(x_0, x - x_0).$$

On the other hand, the derivative is positively homogeneous, hence

$$(1-t)f'(x_0, x - x_0) = f'(x_0, x_t - x_0)$$

and thus by (SVI_M)

$$0 \notin f'(x_0, x_t - x_0)$$

a contradiction. Hence in this case $f_{x_0,x}$ is constant on $[0,1]$ and

$$f'(x_0, x - x_0) = 0^+ f(x_0)$$

In this case, $0 \in \operatorname{int} f'(x_0, x - x_0)$ implies $f(x_0) = Z$, the set valued weak Stampacchia inequality is satisfied for all $x \in \operatorname{dom} f$.

If $x \notin \operatorname{dom} f$, then either $\operatorname{dom} f_{x_0,x} \cap (0,1) = \emptyset$ and $f'(x_0, x - x_0) = \emptyset$, or it exists a $t \in (0,1)$ such that $f(x_t) \neq \emptyset$. In this case, the same argument as above proves the statement, replacing x by x_t. □

Proposition 4.2 *Let $f : X \to \mathcal{G}^\triangle$ be a convex function, $x_0 \in \operatorname{dom} f$. If x_0 solves (svi_M), then it also solves (svi_W) and $f(x) = f(x_0)$ implies that $f_{x_0,x}$ is constant on $[0, 1]$.*

PROOF. Assume $f(x) = f(x_0)$ and $f(x_t) \neq f(x)$ for some $t \in (0, 1)$. By convexity, $f(x_0) \subsetneq f(x_t)$ is satisfied, hence

$$(1-t)\varphi'_{f,z^*}(x_0, x - x_0) = \varphi'_{f,z^*}(x_0, x_t - x_0) \leq \varphi_{f,z^*}(x_t) \dot{-} \varphi_{f,z^*}(x_0) \leq 0$$

is satisfied for all $z^* \in B^*$ and (svi_M) implies the existence of $\bar{z}^* \in B^*$ such that

$$0 < \varphi'_{f,\bar{z}^*}(x_0, x_t - x_0).$$

But this implies $\varphi_{f,\bar{z}^*}(x_0) < \varphi_{f,\bar{z}^*}(x_t)$, a contradiction. Hence in this case $f_{x_0,x}$ is constant on $[0, 1]$ and

$$\forall z^* \in B^* : \quad \varphi_{f,z^*}(x_0) = -\infty \vee \varphi'_{f,z^*}(x_0, x - x_0) = 0.$$

In this case, $\varphi_{f,z^*}(x_0) = -\infty$ for all $z^* \in B^*$ implies $f(x_0) = Z$, the scalarized weak Stampacchia inequality is satisfied for all $x \in \operatorname{dom} f$.

If $x \notin \operatorname{dom} f$, then either $\operatorname{dom} f_{x_0,x} B^* \cap (0, 1) = \emptyset$ and $\varphi'_{f,z^*}(x_0, x - x_0) = +\infty$ for all $z^* \in B^*$, or it exists a $t \in (0, 1)$ such that $f(x_t) \neq \emptyset$. In this case, the same argument as above proves the statement, replacing x by x_t. □

Proposition 4.3 *Let $f : X \to \mathcal{G}^\triangle$ be a convex function, $x_0 \in \operatorname{dom} f$. If x_0 solves (mvi_M), then it solves (mvi_W).*

PROOF. Under the assumption of (mvi_M), let $f(x) = f(x_0)$ be satisfied. By convexity, $\varphi_{f,z^*}(x_t) \leq \varphi_{f,z^*}(x)$ is true for all $z^* \in B^*$ and all $t \in (0, 1)$. Thus either $f_{x_0,x}$ is constant on $[0, 1]$, in which case $f(x_0) = Z$ or $\varphi'_{f,z^*}(x, x_0 - x) = 0$ for all $z^* \in B^*$ with $\varphi_{f,z^*}(x_0) \neq -\infty$, or there exists a $t \in (0, 1)$ such that $f(x_t) \supsetneq f(x)$. In this case, by assumption there exists a $z^* \in B^*$ such that $-\infty \neq \varphi_{f,z^*}(x_t) \leq \varphi_{f,z^*}(x)$ and $\varphi'_{f,z^*}(x_t, x_0 - x_t) < 0$. By convexity of f this implies $\varphi'_{f,z^*}(x, x_0 - x) < 0$. □

Proposition 4.4 *Let $f : X \to \mathcal{G}^\triangle$ be a convex function, $x_0 \in \operatorname{dom} f$. If x_0 solves (MVI_M), then it solves (MVI_W).*

PROOF. Under the assumption of (MVI_M), let $f(x) = f(x_0)$ be satisfied. By convexity, $f(x_t) \preccurlyeq f(x)$ is true for all $t \in (0, 1)$. Thus either $f_{x_0,x}$ is constant on $[0, 1]$, in which case $f(x_0) = Z$ or $f'(x, x_0 - x) = 0^+ f(x) \not\subseteq \operatorname{int} 0^+ f(x)$, or there exists a $t \in (0, 1)$ such that $f(x_t) \supsetneq f(x)$. In this case, by assumption $f'(x_t, x_0 - x_t) \not\subseteq 0^+ f(x_t)$. Let $s \in (0, 1)$, then

$$\begin{aligned} f'(x, x_0 - x) &\supseteq \frac{1}{s + t - st} \left(f(x_t + s(x_0 - x_t)) \dot{-} f(x) \right) \\ &\supseteq \frac{1}{s + t - st} \left(\left(f(x_t + s(x_0 - x_t)) \dot{-} f(x_t) \right) \oplus \left(f(x_t) \dot{-} f(x) \right) \right) \end{aligned}$$

By assumption, $f(x) \subseteq f(x_t)$, hence

$$0^+ f(x_t) \subseteq \left(f(x_t) \dot{-} f(x)\right),$$

which implies

$$\begin{aligned} &\left(f(x_t + s(x_0 - x_t)) \dot{-} f(x_t)\right) \oplus \left(f(x_t) \dot{-} f(x)\right) \\ \supseteq &\left(f(x_t + s(x_0 - x_t)) \dot{-} f(x_t)\right) \oplus 0^+ f(x_t) \\ = &f(x_t + s(x_0 - x_t)) \dot{-} f(x_t) \end{aligned}$$

and therefore

$$f'(x, x_0 - x) \supseteq \frac{s}{s + t - st} \left(\frac{1}{s} f(x_t + s(x_0 - x_t)) \dot{-} f(x_t) \right).$$

Moreover, $f'(x_t, x_0 - x_t) \not\subseteq 0^+ f(x_t)$, hence choosing $s \in (0, 1)$ small enough,

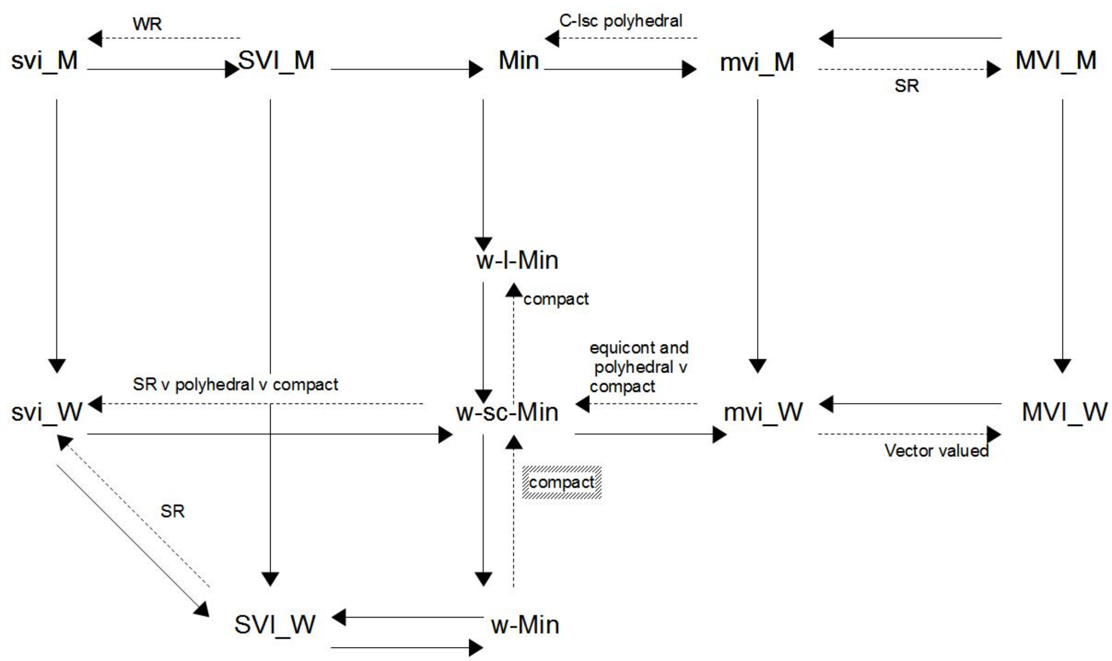
$$\frac{1}{s} \left(f(x_t + s(x_0 - x_t)) \dot{-} f(x_t)\right) \not\subseteq 0^+ f(x)$$

is satisfied, proving

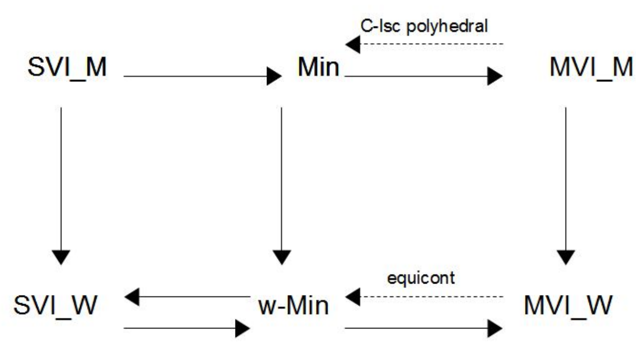
$$f'(x, x_0 - x) \not\subseteq \frac{s}{s + t - st} 0^+ f(x),$$

for some $s \in (0, 1)$. Thus $f'(x, x_0 - x) \not\subseteq \operatorname{int} 0^+ f(x)$, as desired. $\square$

Overall, we provided the following implications.



If additionally $f = \psi^C$ is assumed, i.e. f is the epigraphical extension of a vector valued function, then the solution sets of scalarized and set valued versions of each variational inequality coincide and the following scheme represents the implications proven.



We remark that the latter scheme is a straightforward extension of the scheme of relations originally provided by Giannessi for his vector variational inequalities in vector optimization. Therefore, as an application, we have proved that set–optimization approach provides a useful tool to study vector optimization, by considering the epigraphical extension of the objective function. The main advantage we see in this approach is to work in an order complete space, rather than a partially ordered space.

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